Chaos Theory and the Science of Fractals, and their Application in Risk Management

"To those of the younger generation who resist the confident blandishments of their teachers: in them resides the hope of future economics"

Miroswki (2004)

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Winter – 2009
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EXECUTIVE SUMMARY

Before Chaos Theory consolidated as a main paradigm in science many preconceived ideas had to be modified, in particular, the Newtonian mechanistic perspective of the world characterized by rigid assumptions, mathematical formalism and methodological reductionism. Nevertheless, this change brought great progress for scientific research, as it opened the opportunity to explore the complexity and roughness in natural systems. Unfortunately, financial theories have not evolved at the same pace. Current financial paradigms, based on Neoclassical postulates, are still linked to Newtonian scientific thinking. This has lead financists to address current complexity of financial markets with an inadequate language and method.

Therefore, in this investigation, it is proposed to adopt the foundations of Chaos Theory and the Science of Fractals to explain financial phenomena. This will imply a change in the neoclassical notions of rationality, perfect markets and equilibrium models, and the mathematical assumptions of smoothness, continuity and symmetry. With the emergence of this new theory, thus, it would be possible to describe the messiness of today’s financial markets. The key here is to understand the fractal characteristic of the market, as it provides the adequate perspective and mathematical tools to analyze it.

Consequently, financial theory will benefit from Chaos Theory and the Science of Fractals in that they will provide more adequate assumptions, and hence, more realistic models of financial behavior. This will be particular important for risk management, as it would allow professionals in this area to understand risk in a more comprehensive manner. Moreover, with the use of fractal statistics, it would be possible to improve financial risk models. To illustrate this point, it would be shown how adopting the hypothesis of this theory in Value-at-Risk, the de facto measure of market risk, may contribute to the enhancement of risk assessment, and even, regulation.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>4</td>
</tr>
<tr>
<td>Thesis Statement</td>
<td>5</td>
</tr>
<tr>
<td>Structure</td>
<td>5</td>
</tr>
<tr>
<td>Literature Review/Method</td>
<td>7</td>
</tr>
<tr>
<td>Limitation</td>
<td>9</td>
</tr>
<tr>
<td><strong>LITERATURE REVIEW</strong></td>
<td></td>
</tr>
<tr>
<td>Classical Mechanics</td>
<td>9</td>
</tr>
<tr>
<td>Second Law of Thermodynamics</td>
<td>12</td>
</tr>
<tr>
<td>Quantum Mechanics</td>
<td>12</td>
</tr>
<tr>
<td>Chaos Theory</td>
<td>14</td>
</tr>
<tr>
<td>Fractal Geometry</td>
<td>17</td>
</tr>
<tr>
<td><strong>CONCLUSION</strong></td>
<td>20</td>
</tr>
</tbody>
</table>

## PART I. FROM NEOCLASSIC ECONOMICS TO CHAOS THEORY AND THE SCIENCE OF FRACTALS

**Neoclassical Theory**

- Neoclassical Assumptions                                              21
- Efficient Market Hypothesis and Random Walk Theory                   22
- Failures and Inconsistencies of Neoclassical Theory:
  - Discrepancy with economic reality                                  25
  - Newton’s assumptions in Neoclassical Theory                       27
  - Newton’s mathematics and method in Neoclassical Theory            28

**Applying Chaos Theory and the Science of Fractals in Economics and Finance**

- Complexity Economics                                                30
- Chaos Theory and the Science of Fractals in Finance:
  - From the Efficient Market Hypothesis to the Fractal Market        33
    - Hypothesis
    - From a Random Walk to a Fractal Approach to the Market
      - A Fractal Financial Market                                      36
      - From Normal Distribution to Stable Paretian Distributions       37
      - From a Random Walk to a Multifractal Analysis                   41

## PART II. IMPLICATIONS IN RISK MANAGEMENT

**I. Towards Better Quantitative Models in Risk Management**

- Implication of a fractal market for risk management                  46
- Value-at-Risk in a stable Paretian market:
  - The rise of VaR                                                    47
  - The risk at Value-at-Risk                                          49

**II. A New Set of Qualitative Tools for Risk Managers**

- Perspective on Regulation                                            52

**CONCLUSION**

- References                                                           60
- Appendix I. The Hurst Exponent                                       69
- Appendix II. Multifractal Analysis                                   72
- Appendix III. Calculation of the Variance-Covariance VaR            74
- Appendix IV. Rachev et al (2003)                                     76
INTRODUCTION

The evolution of the financial and business environment since 1970 has come with a very high price. Without clear warnings, at least eight major crises have struck the financial sector. The current economic turmoil, caused by the subprime crisis in the United States, can be considered one of the worst financial disasters. For the first time, problems that originated in one country had global effects leading the world economy to a serious recession. However, it is in these critical times that beliefs are reevaluated and new paradigms emerge to guarantee that next time crises come at a discounted value.

Indeed, one of the mayor lessons of the subprime crisis has been that current financial models are not based on adequate assumptions. Evidently, Neoclassical theory, today’s mainstream economic and financial paradigm, has become obsolete for explaining the complexity of financial markets. It oversimplifies reality, and therefore, can only address problems under ideal or normal conditions. This is especially problematic for risk management, as its objective is to ensure that risks are controlled even in the wildest financial moments. For this reason, it is necessary to look for alternative theories that allow the description of real market dynamics, and provide accurate tools to measure the apparent disorder of today’s capital markets.

Consequently, this investigation proposes the application of Chaos Theory and the Science of Fractals to finance. Within this new framework, it would be possible to find a more general, but coherent perspective to study financial phenomena. Mainly, it focuses on the fractal structure of capital markets to be able to develop new analytical and mathematical tools. This would allow financists to move upwards in the understanding of financial behavior.

However, as Leon Tolstoi one said: “I know that most men, including those at ease with problems of the greatest complexity, can seldom accept even the simplest and most obvious truth if it be such as would oblige them to admit the falsity of conclusions which they have explain to colleagues, taught to others and woven, thread by thread, into the fabric of their lives” (Chorafas 1994 p 69). Hopefully, financists turn their attention to this new proposition so that theories in this field can finally reflect the thinking of the XXI century.
THESIS STATEMENT

The aim of this investigation is to study Chaos Theory and the Science of Fractals in finance, and its implications to risk management. In order to fulfill this purpose the following questions will be answered:

- What are Chaos Theory and the Science of Fractals?
- What can Chaos Theory and the Science of Fractals contribute to financial theory?
- What are the differences between this new emergent framework and mainstream theories in economics and finance?
- What would be the implications of adopting the postulates of Chaos Theory and The Science of Fractals for risk management?

STRUCTURE

This thesis is divided into three main parts. Part I briefly summarizes the evolution of physics from the time of Newton to the recent advancements in Chaos Theory and the Science of Fractals. The purpose of this section is to describe the progress that science has experienced, and to explain the core notions of Chaos Theory and fractals.

Part II contains two sections. The first section explores the theoretical framework of mainstream economic theory, Neoclassical Theory, which includes the Efficient Market Hypothesis and Random Walk Theory. Afterwards, the similarities between Newton's physics and Neoclassical Theory are analyzed. The second section studies the application of Chaos Theory in economics and finance. Initially, it explains complexity economics, as the economic discourse of Chaos Theory. After that, it proceeds to introduce Chaos Theory and the Science of Fractals in financial paradigms. This will lead to the presentation of a new theoretical framework, known as the Fractal Market Hypothesis, and new mathematical tools represented in fractal statistics.

Part III is also divided in two sections. The first one attempts to highlight the relevance of Chaos theory and the Science of Fractals in the quantitative models of risk management. To illustrate this point, assumptions of this new theory are applied to Value-at-risk, one of the most important measures of risk. The second section contains the importance of this new paradigm in renovating the qualitative tools of risk managers. At the end, a short perspective in regulation is given.
The following figure illustrates the structure of this study:

   - Second Law of Thermodynamics
   - Quantum Mechanics
   - CHAOS THEORY
     - Science of Fractals

2. Neoclassical Theory
   - Efficient Market
   - Random Walk Theory
   - Complexity Economics
   - Fractal Market
   - Fractal Financial Market

3. Implications for Risk Management
This thesis has a theoretical approach, thus its objective is to review existing literature on Chaos Theory and the Science of Fractals, and its applications to economics and finance. In the literature collection of Part I, many scientific texts that deal with Classical mechanics, Second Law of thermodynamics and Quantum mechanics are reviewed. A primary source in this topic is the book of Rosenblum and Kuttner (2006), which clearly describes all the scientific discoveries that occurred from Newton’s time to the quantum field. On the other hand, for the explanation of Chaos Theory, the information is gathered from many distinctive sources including scientific, philosophical, computational, organizational and economic texts. The reason is that Chaos Theory is applied to a number of different fields, and each one of them has contributed to its development. Therefore, to have a holistic perspective, it is necessary to address this theory from various angles. In the case of fractal geometry, on the contrary, the documentation is based on books of Benoit Mandelbrot (1997, 2004 and 2005), its inventor.

In Part II, the literature review focuses on the developers of the most significant theories in economics and finance. Therefore, to examine Neoclassical Theory, authors such as Louise Bachelier (1900), Eugene Fama (1965 and 1970) and Cootner (1964) are studied. For the link between Neoclassical theory and Newtonian physics the main source is Mirosławi (2002 and 2004), although other authors make important contributions. With regards to the application of Chaos Theory in economic theory, this thesis refers to Rosser (2004), as this author reunites in his book texts dating from 1960 until now that explain complexity thinking in economics. Finally, there are two main sources in the study of Chaos Theory and the Science of Fractals in finance. For the theoretical framework, this thesis explains the proposition of Peters (1994); and for the mathematical concepts it reviews the ideas of Benoit Mandelbrot (1963, 1997, 2003, 2004 and 2005). Nevertheless, because early texts still lack of proper development and vagueness in some concepts, the explanation of his main hypothesis are found in the most recent books. In particular, Mandelbrot (2004) gives an adequate and comprehensive synthesis of his ideas.

In the final section, the concepts behind the VaR measure and regulation are provided by many sources. However an important contributor is Jorion (2007). Finally, the empirical evidence of non-Gaussian distributions is taken from Rachev (2003), one of the most complete investigations on this topic. Nevertheless, it is important to highlight that because this work presents new concepts for the author, and perhaps the audience, many complementary texts are included through the entire thesis.
LIMITATIONS

Nowadays, Chaos Theory and the Science of Fractals encompass multiple issues in different fields. Therefore, an exploration of this subject is complicated, as there is not a unique framework that contains its main postulates. Furthermore, because this theory is still developing, many ideas continue to emerge, and hence concepts are redefined or complemented continuously. Indeed, today scientists are trying to link Chaos Theory with other scientific disciplines to establish a more general theory. Therefore, this investigation will try to reunite the most important ideas that are relevant to economics and finance. However, in the future it would be possible to address more topics. The same occurs with complexity economics, the economic discourse of Chaos Theory. As a consequence, the ideas exposed in this work, are just a part of the vast collection that this school of thought has produced.

The application of Chaos Theory and the Science of Fractals in finance presents a different problem. While physics and economics have already explored this field, in finance it is still in its infancy stages. Consequently, there is vagueness in certain concepts, missing links and need of consensus. Thus, this investigation is limited to the existent literature, which still needs to be further developed. Additionally, in the presentation of the new mathematical techniques, the main disadvantage is the lack of empirical evidence. As a result, this thesis will just present the arguments, but will not go into a deeper analysis of their empirical validity.

Regarding the implications of Chaos Theory and the Science of Fractals to risk management, this thesis will just deal with the variance-covariance technique of Value-at-Risk to illustrate the change in assumptions in risk models. The reason is that being such an important risk measure, it is the method that has accumulated more empirical evidence, and therefore, it is possible to draw conclusions based on real studies.

Having acknowledged this, it is important to highlight that the conclusions of this thesis are not definite. There are various issues that still need to be explored.
LITERATURE REVIEW: THE DEVELOPMENT OF CHAOS THEORY IN SCIENCE

From the seventeenth century to the nineteenth century, Newtonian reductionism and mechanistic view of the world predominated in the scientific discourse and approach to science. However, discoveries in thermodynamics and quantum mechanics led scientists to question existing ideas, and to embrace a more organic view of nature, less orderly and less predictable. This contributed to the emergence of a new paradigm in science, Chaos theory and the Science of Fractals. Within this new framework, science has found the adequate idiom and tools to better understand our complex physical world.

I. Classical Mechanics

Classical Mechanics begins with Isaac Newton, one of the greatest thinkers of the seventeenth century (the Age of Reason or Age of Rationalism). In 1687, Newton published his book *Philosophiæ Naturalis Principia Mathematica*, which included the laws of motion and the universal force of gravitation. These propositions were unprecedented for science as Newton explained with a comprehensive mathematical reasoning how terrestrial and celestial physics could be described by just one set of rules. Newton’s ideas became then axioms for classical mechanics and even played a pivotal role in the birth of modern science. To this day, academics are still influenced by Newton’s intellectual contributions and scientific method.

Beneath Newton’s grand postulates there are two implicit assumptions. First, Newton’s laws are “immutable” and independent in time, in the sense that past and future play the same role. Therefore, if initial conditions are known, it is possible to retrodict events in the past and to predict them into the future. Secondly, they are universally deterministic, since they

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1 The seventeenth century is known as the Age of reason because almost all the great philosophers of the time were trying to introduce the rigour of mathematics into all areas of knowledge.
2 The law of motion comprises three main postulates: 1) every object is at a state of rest unless some external force acts upon it changing its uniform motion; 2) the object will move in the same direction and proportional to the force applied \[\text{Force} = \text{Mass} \times \text{Acceleration}\]; and 3) every action has an equal and opposite reaction. Thus if the body 1 exerts on body 2 a Force A, then body 2 exerts on body 1 a Force B, resulting in \[F(A) = F(B)\].
3 The law of gravitation is a special case of the second postulate. This law states that there is an attraction between all bodies with mass, which is directly proportional to the quantity of matter that they contain, and inversely proportional to the square of the distance between them.
4 Before Newton there was still Aristotle’s strong division between earthly and cosmic phenomena. In Aristotle’s opinion, it was only the heavenly world that could be described with a mathematical language; whereas on earth, Aristotle believed in theology and natural diversity.
5 Determinism means that every event is the inevitable result of a preceding occurrence.
perceive nature as an intelligible system governed by the principle of causality. Thus, in the Newtonian perspective, systems work in a reasonable and predetermined manner similar to a “clockwork machine”\(^6\). As Lavoie (1989) explains, “[in Newtonian physics] nature is seen to be outgrowth of predictable laws that are not the design of any conscious entity, but in principle are subject to human mastery” (Lavoie 1989 p 54).

This mechanical interpretation of the universe is observed in both Newton’s discourse and approach to science. As the inventor of the calculus, Newton was able to simplify natural phenomena into linear equations and other predictable formulas. Even more, his theorems followed a mathematical logical system similar to the method of Euclidean geometry\(^7\), where few elegant simple premises, combined with a deductive logic, revealed the “truth” of the universe. Newtonian laws are, therefore, characterized by their mathematical formalism and methodological reductionism.

For more than a century, Newton’s prepositions were considered the ultimate science and its mathematics the ultimate expression of reality. Followers of Newton “were more sure of Newton’s postulates than Newton was—until 1800” (Rosenblum and Kuttner, 2006, pp 40). In the nineteenth century discoveries in the cosmic and natural world showed that the universe was in fact govern by more complex processes than those observed by classical mechanics (see Figure 1.1). Consequently, physicists modified their dialogue and methodology allowing the coherent development of new theories. The deterioration of the Newtonian paradigm’ supremacy was therefore inevitable in the evolution of science.

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\(^6\) Inspired by Newton’s ideas, LaPlace once affirmed: “Give me only the equation of motion, and I will show you the future of the Universe” (in Hsieh and Ye 1991 p 8). This statement is known as LaPlacean deterministic predictability.

\(^7\) Euclidean Geometry is a mathematical system named after the Greek mathematician Euclid of Alexandria. In his famous book *Elements*, Euclid organized and summarized the geometry of ancient Greece based on lines, planes and spheres. In his discussion about geometry, thus, he describes the smoothness and symmetry of nature in idealized or Platonic terms. However, Euclid’s great contribution was his use of a deductive system for the presentation of mathematics. As so, he explained a series of theorems all derived from a small number of axioms. Although Euclid’s system no longer satisfies modern requirements of logical rigor, its importance in influencing the direction and method of the development of mathematics is undeniable. Indeed, Euclidean Geometry remained unchallenged until the middle or late nineteenth century.
{1687 – Newton: Law of Motion and universal gravitation (basis for classical physics)}

1824 – Carnot made the first statement of what later became the Second Law of thermodynamics


1850 – Rudolf Clausius: Second Law of Thermodynamics

1859 – Charles Darwin and Alfred Wallace: Theory of Evolution and natural selection

1873 – James Clerk Maxwell: Theory of Electromagnetism

1877 – Ludwig Botzmann: Statistical Definition of entropy

1900 – Max Planck: Planck’s Law of black body radiation and Planck's constant (basis for quantum mechanics)

1905 – Albert Einstein: Theory of Special Relativity (explanation of Brownian Motion by kinetic theory)

1913 – Niels Bohr: Model of the atom (quantum theory of atomic orbits)

1915 – Albert Einstein: Theory of General Relativity

1925 – Erwin Schrodinger: Schrodinger equation (quantum mechanics)

1927 – Werner Heisenberg: Uncertainty Principle

Niels Bohr: Principle of Complementarity and Copenhagen interpretation of Quantum Mechanics

1928 – Paul Dirac: Dirac equation (quantum mechanics)

Figure 1.1: Timeline with the most important discoveries in physics of the 19th and 20th centuries.
II. **The Second Law of Thermodynamics**

In 1824, Sadi Carnot published the first statement of what later developed into the Second Law of Thermodynamics. Carnot argued that heat could not spontaneously flow from cold objects to hot ones. More specifically, heat could flow from a higher to a lower temperature, but never in reverse direction, except with the action of an external force. As opposed to classic mechanics, this demonstrated that some processes are simply irreversible in time. Indeed, “if there were no second law, the universe would be like a giant clock that never run down” (Lightman 2000 p 63). This is the first rupture with classical thinking, more specifically, with the Newtonian conception about time.

By 1850, Rudolf Julius Clausius gave the classic perspective of both laws of thermodynamics. The first one, known as the Conservation Principle, states that energy in the universe remains constant; and the second one states that the universe’ entropy moves towards a maximum level. In other words, there is tendency for matter to become more disordered with the progression of time.

The Second Law of thermodynamics has important implications. As mentioned before, it clearly distinguishes the past from the future, arguing that some processes can never return to their initial conditions. This is known as the “arrow of time”, which emphasizes the construction role of irreversibility in the evolution of a system. Secondly, it demonstrates that the universe tends to a more disorganized state. As a result, while classical science emphasized stability, equilibrium and determinism, the Second Law of Thermodynamics describes instabilities, non-equilibrium dynamics and evolutionary trends.

III. **Quantum mechanics**

The second rupture comes from the development of quantum mechanics, which will dramatically change the perception of the world and the role of science. From the end of the nineteenth and twentieth century, experiments in the atomic world, in particular the study of the properties of light, demonstrated that classical science was not able to explain certain behaviour of particles. Although classic mechanics made a good approximation of the

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8 Thermodynamics is a branch of physics that deals with the various phenomena of energy and related properties of matter, especially the laws of transformation of heat from one form to another. The basis of this science is experimental observation.

9 As it is observed, the First Law of Thermodynamics is still under the Newtonian perspective of the world.

10 Originally entropy was used to describe the amount of energy unavailable for useful work in a system that was going through a change. However later it was associated with the degree of randomness or disorder present in any given construct.

11 The term quantum comes from Max Planck’s suggestion in 1900 that radiant energy was emitted from a heated body in discontinuous portions that he termed quanta. Afterwards, the term quantum mechanics was generalized to define a branch of physics that studies the physical reality at the atomic level of matter.
behaviour of objects at a large scale, it encountered problems when studying the microscopic phenomena. Within the framework of quantum mechanics, scientists then developed new empirical theories that accurately describe all the “details” of the physical world. Nowadays, classical physics are considered a specific case of quantum mechanics, while quantum mechanics are regarded as “the most accurate theory in all science” (Rosenblum and Kuttner 2006 p 82).

The main contribution of quantum mechanics is that it changes the deterministic Newtonian perspective and gives science a natural uncertainty that can only be described by states of probability. For instance, the German physicist Heisenberg proposed the uncertainty principle as he discovered that is not possible to determine an exact position and moment of an object simultaneously. More generally, this principle argues that it is unlikely to know with precision the values of all the properties of a system at the same time. In this way, the properties that are not possible to be described can only be inferred by probabilities. At last, even in exact sciences such as mathematics there is not total certainty.12

Moreover, Niels Bohr, a Danish physicist, developed the concept of complementary, which states that subatomic particles have simultaneous wave and particle properties. However, because it is not possible to observe both states at the same time, it is the observer who determines the properties of an object. This implies that measuring instruments affect the behavior of atomic objects and the conditions under which certain phenomena appear. In 1927 Niels Bohr wrote: "Anyone who is not shocked by quantum theory does not understand it." (Rosenblum and Kuttner 2006 p 52). Together, the complementary and uncertainty principles lead to an essential ambiguity in science, where nothing is certain, nothing is complete, even more nothing is real unless it is perceived.13

This change in the notion of reality inevitably was accompanied with a modification in the role of science. With the Copenhagen Interpretation14 of quantum mechanics, this discipline is redefined as attempts to explain the different physical phenomena with mathematical formulations and empirical observations that do not go beyond of what the evidence suggests. Consequently, it proposes the integration of different scientific disciplines to be able to give a more coherent explanation of our observations. A holistic and anti-reductionist approach of science, hereby, is necessary to discover the so-called “universal truths”.

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12 Kurt Gödel will complement this idea with his incompleteness theorem. In 1931 Gödel demonstrated that there were limits in mathematics, because there are problems that do not have an established solution. In other words, for a given group of postulates there is always going to be one of them whose validity can neither be proven nor disproven.

13 As the Irish Bishop George Berkeley said: “Esse est percipi” (To be is to be perceived)

14 The Copenhagen Interpretation consists on a philosophical construct developed to understand the atomic world. The founding father was Niels Bohr, but other scientists such Werner Heisenberg and Max Born made important contributions.
Evidently, quantum mechanics contradicts the Newtonian clockwork machine notion. With this “new science”\(^\text{15}\), “the universe begins to look more like a great thought than a great machine” (Rosenblum and Kuttner 2006 p 51). Nevertheless, accepting the quantum theory means confronting a big enigma as it postulates ultimate randomness in nature.

IV. Chaos Theory

Chaos Theory is the final rupture with Newtonian mechanics. This theory studies systems that appear to follow a random behavior, but indeed are part of a deterministic process. Its random nature is given by their characteristic sensitivity to initial conditions that drives the system to unpredictable dynamics. However, in a chaotic system, this non-linear behavior is always limited by a higher deterministic structure. For this reason, there is always an underlying order in the apparent random dynamics. In Sardar and Abrams (2005) words: “In Chaos there is order, and in order there lies chaos” (Sardar and Abrams 2005 p 18).

Kellert (1993) defines chaos theory as “the qualitative study of unstable aperiodic\(^\text{16}\) behaviour in deterministic nonlinear\(^\text{17}\) dynamical systems” (in Mc Bride 2005 p 235). Chaotic systems are said to be mathematically deterministic because if the initial measurements were certain it would be possible to derive the end point of their trajectories. Nevertheless, contrary to Newtonian science, Chaos theory deals with nonlinear feedback forces\(^\text{18}\) with multiple cause and effect relationships that can produce unexpected results. Appropriately, a chaotic system cannot be understood by the simple disintegration of the whole into smaller parts.

The investigation on this field can be trace to the early 1900’s when the physicist Henri Poincare was searching for a mathematical explanation of planetary movement. Before Poincare’s discoveries in astronomy, the mathematical orbit equations followed Newton’s laws, and thus, were completely deterministic. Therefore, it was believed that if methods to measure initial conditions were improved, it was possible to obtain a more accurate prediction of the planet’s trajectories\(^\text{19}\). Contrary to this idea, Poincare observed that a very small difference in the starting positions and velocities of the planets could actually grow to an enormous effect in the later motion. This was a proof that even if initial measurements

\(\text{15}\) The “new” science is a general term for all the theories and ideas generated in different academic disciplines that do not correspond to the “classical” scientific explanation. Quantum mechanics is part of this approach to science, with other disciplines and theories, which leave behind the idea of the Newtonian world to model our complex reality

\(\text{16}\) An aperiodic system is the one that does not experience a completely regular repetition of values. However the variables associated with the system are limited by a fixed and definable space.

\(\text{17}\) The significance of the concept of nonlinearity is that the whole is greater that or less than the sum of its parts. By contrast in a linear world, the effects are always proportional to their causes.

\(\text{18}\) Feedback is present in a system when the inputs affect the outcome, and at the same time, the outcome will influence the inputs.

\(\text{19}\) In 1894 Lord Kevin said: “there is nothing to be discovered in physics now. All that remains are more and more precise measurements” (Rosenblum and Kuttner (2006) pp 39).
could be specified with high precision, the uncertainty would still remain huge in certain systems. Poincare’s discovery was neglected for many decades as it clearly contradicted the mechanistic Newtonian perspective.

However, in the early 1960’s a small part of the scientific community became simple dissatisfied with existing paradigms. They often ignored important aspects to uphold linear equations and a reductionist method. It became then apparent that Newtonian mechanics had serious limitations in explaining the complexity of the universe. Therefore, scientists started looking for new explanations and new approaches that were more coherent with the organic nature of the world. The advent of computers facilitated this task, and thus the scientific community progressively turned to the study of non-linear dynamics, patterns, and other complex behavior that was excluded in classical science.

A significant progress in the emergence of the new science came with Edward Lorenz’s proposition of the “butterfly effect”. In his meteorological investigations, Lorenz found that variations in the decimals of initial measurements of the weather predicted a completely different motion. To illustrate better his theory, Lorenz gave the example of how a butterfly that flags its wings in Brazil can cause a tornado in Texas, indicating that a small wind could change the weather in few weeks. This is known as the butterfly effect, the “signature” of chaos theory, and it makes the point that some systems are highly sensitive to insignificant changes in the starting conditions.

Gradually, physicists discovered that most natural systems are characterized by local randomness as well as global determinism. These two states can coexist, because randomness induces the local innovation and variety, but determinism gives the global structure. Therefore, a random system behaves always within certain bounds. Fractal structures are clear examples of this type of systems. As Peters (1994) said:

“Now we have the third blow to Newtonian determinism: the science of chaos and fractals, where chance and necessity coexist. In these systems, entropy is high but never reaches maximum disorderly states because of global determinism. Chaotic systems export their entropy, or “dissipate” it, in much the way that mechanical devices dissipate some other energy as friction. Thus, chaotic systems are also dissipative and have many characteristics in common with thermodynamics – especially the arrow of time” (Peters 1994 p 9).

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20 A fractal is a rough or fragmented geometric shape that can be split into equal parts, each of which is resembles the whole. The term “fractal” was coined by Benoit Mandelbrot and was derived from the Latin word fractus, which means “broken” or “fractured”. This concept will be further elaborated in the following section, as it is key to understand chaotic structures.

21 A dissipative system is characterized by having order through fluctuations, phase transitions, and multiple bifurcations. Prigogine, pioneer of non-classical mechanics, said “the appearance of
In fact, the "cycle" of a chaotic system is also full of fluctuations and bifurcations. This chaotic process is characterized by the initial conditions and by the system’s strange attractors\(^{22}\). These two components will provide the global structure, and thus, will determine the non-linear progression of the system over time. In the other hand, the random motion will be given by the influence of internal or external forces\(^{23}\) that will change the trajectories of the system arbitrarily. As a result, a chaotic system will exhibit different phases of stability and unstable states passing through certain critical points known as the “edge of chaos”\(^{24}\). At these particular stages, the system is suspended in a transitional position between stability and disorder. Afterwards, it will experience a bifurcation, as new emergent patterns arise and trajectories move towards a new strange attractor. If the system finally reaches another stable state, a change to the previous attractor will not be possible\(^{25}\). Furthermore, it would be unfeasible to predict at a specific time when the next change will occur due to the fact that the time between bifurcations is aperiodic.

The “sand pile model” can better illustrate this self-organized criticality. Initially, the equilibrium conditions of the model will be given by a flat structure. However, as randomly grains of sand start building up the pile, it will grow away from its long-term equilibrium. When the sand pile is big enough, an avalanche will restructure the sand pile as if a new state of self-critically emerged. This type of chaotic dynamics can actually explain the motion of molecules to the behaviour of economic agents.

Overall, the objective of Chaos theory is to study changing environments, full of nonlinear dynamics, discontinuities, feedback systems and intelligible, but not certain, patterns. This dissipative structures occurs at bifurcation points where new solutions of non linear equations of evolution become stable (…) At bifurcations, there are general many possibilities open to the system, out of which one is randomly realized. As a result, determinism breaks down, even on macroscopic scale” (Prigogine 2005 p 65). In this non-equilibrium dynamics the concept of self-organization emerges, as the process that maintains order and allows systems to adapt to the environment. Prigogine named these structures dissipative, because they require more energy to sustain themselves. In general, dissipative structures involve some damping process, like friction.

\(^{22}\) Attractors "represent the state to which a system settles, depending on the properties of the system" (Sardar and Abrams 2005 p 47). A chaotic system possesses "strange" attractors because they reconcile two contradictory effects. In one hand, they are attractors thus trajectories of the system converge to them. And in the other, the dependence of the system on initial conditions will make those trajectories deviate rapidly to another attractor. Here, it is important to highlight that while strange attractors exist in an infinite dimensional space, called phase space, they themselves have only finite dimensions. This is the reason why strange attractors are described as fractal objects.

\(^{23}\) In a chaotic process the change may be created by an endogenous process, as well as exogenous one, contrary to Newtonian mechanics where internal dynamics are not taken into account.

\(^{24}\) This concept “involves a specific form of the more general idea of structures emerging from low level interactions, with the argument that this higher order self-organization emergence is most likely to occur in certain zones of the system, the borderline between rigid order and total disorder” (Rosser 1999 p 17).

\(^{25}\) This is where irreversibility of time plays an important construction role. It is possible that the following bifurcation involves a shift back to the previous attractor, but chaotic systems never return to the same exact state.
influenced deeply the discourse of science, allowing it to move upwards to a better understanding of the physical world. Most important, it triggered a significant change in its methodology. According to Mirowski (2004):

“The breakthrough came when physicists stopped looking for deterministic invariants and began looking at geometric patterns in phase space. What they found was a wholly different kind of order amid the chaos, the phenomenon of self-similarity at different geometric scales. This suggested that many phase-space portraits of dynamical systems exhibited fractal geometry; and this in turn was taken as an indication that a wholly different approach must be taken to describing the evolution of mechanical systems” (Mirowski 2004 p 243)

Chaos theory, therefore, proposes the science of fractals as the framework where new tools can be found and new ways of solving problems are explored.

V. Fractal Geometry

Benoit Mandelbrot developed the field of fractal geometry between 1970 and 1980 with books such as *Fractals: Forms, chance and dimensions* (1977) and *Fractal Geometry of Nature* (1982). The importance of this science is that objects are not reduced to a few perfect symmetrical shapes as in Euclidean geometry. Instead it explores the asymmetry, roughness\textsuperscript{26}, and fractal structures of nature. With fractal geometry: “clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lighting travel in straight lines” (Mandelbrot 2004 p 124). In this way, as Euclidean Geometry served as a descriptive language for the classical mechanics of motion, fractal geometry is being used for the patterns produced by chaos.

A fractal is a shape made of parts similar to the whole in some way, thus they look (approximately) the same whatever scale they are observed. When fractals scale up or down by the same amount, they are said to be self-similar. In the contrary, if they scale more in one direction than another, they are called self-affine. And in their most complex form, they are named multifractals, which scale in many different dimensions and ways\textsuperscript{27}. This diversity of fractals allows them to be found in concrete shapes such as cauliflowers and even in statistical patterns, such as the ones found in financial data.

\textsuperscript{26} For Mandelbrot, ROUGHNESS is no mere imperfection from some ideal, but indeed the very essence of many natural objects.

\textsuperscript{27} This variability is the reason that the prefix "multi-" was added to the word "fractal."
The concept of fractals is inextricably connected to the notion of fractal dimension. In Euclidean mathematics a point had one dimension, a line two and a cube three. With Einstein, and his Relativity theory, the physics world added time as the fourth dimension. However in fractal science, the dimension depends on the point of view of the observer. “The same object can have more than one dimension, depending on how you measure it and what you can do with it. A dimension needs not to be a whole number; it can be fractional. Now and ancient concept, dimension, becomes thoroughly modern” (Mandelbrot 2004 p 129).

The fractal dimension is important because it recognizes that a process can be somewhere between deterministic or random.

In spite of this, fractal geometry is in fact a simplifying and logical tool. In mathematics, fractal functions work like chaotic systems where random changes in the starting values can modify the value of the function in unpredictable ways within the system boundaries. The famous Mandelbrot set demonstrates this connection between fractals and chaos theory, as from a

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The fractal dimension gives a qualitative measure of the degree of roughness, brokenness or irregularity of a fractal.

This closely resembles the perspective of quantum mechanics, including the importance of probability theory to map the unknown. It is important to highlight that the probability distribution characteristic of fractals is hyperbolic, just as one might say the probability distribution of the Euclidean world is Gaussian.
very simple mathematical feedback equation, highly complex results are produced (see Figure 1.4) 30.

The key to understand fractals is then to discern those fundamental properties that do not change from one object under study to another 31. Even more, by studying the fractal structure of chaotic systems, it may be possible to determine the critical points where predictability in a system breaks down. Accordingly, “fractal geometry is about spotting repeating patterns, analyze them, quantify them and manipulate them, it is a tool of both analysis and synthesis”. (Mandelbrot 2004 p 126).

The purpose of fractal geometry, therefore, is not to obscure the role of science; but instead to give a more creative approach to scientific method. While classical physics used the top down Euclidean method, the fractal approach grows from the bottom up, more specifically from observation. Even more, it is a parsimonious method as from simple rules complex phenomena can be explained. According to Mandelbrot (2004), “[The aim of science is] to provide a compromise between two very different goals: satisfactory statistical fit to observations and the highest achievable parsimony” (Mandelbrot 2004 p 179-180). Given these special features, Fractal Geometry has extended to areas such as hydrology, meteorology and geology, and even to economics.

30 After all, “in the hands of a mathematician even the most trivial random process can generate surprising complexity and highly structured behavior (...) Andrei Nikolaievitch Kolmogorov, Russian mathematician founder of modern probability theory wrote: ‘the epistemological value of probability theory is based on the fact that chance phenomena, considered collectively and in a grand scale, create a non-random regularity’” (Mandelbrot 2004 p 30).

31 In Mandelbrot’s words, “the key is to spot the regularity inside the irregular, the pattern in the formless” (Mandelbrot 2004 p 125).
CONCLUSION

The evolution of complexity theory show us how difficult is to swim against preconceived ideas. The mechanistic perspective, with its fixed theories, linear methods and simple cause and effect approach, was part of the scientific discourse for more than one hundred years. The first consistent rupture was made in the area of thermodynamics, and later in the quantum mechanics. Even Darwin’s evolution theory and Einstein’s relativism help to set to ground for the emergence of a new synthesis in science. The result was the “antithesis” of the Newtonian paradigm, Chaos theory, which proposed a new language and tools to address our complex reality. As Gleick famously said “where Chaos begins, classical science stops” (Gleick, 1993).
PART I. FROM NEOCLASSICAL ECONOMICS TO CHAOS THEORY

Historically, it has been observed that the ideas that predominate in science influence the economic paradigms of the same period of time. Thus, the legacy of Newtonian mechanics is observed in the economic ideas of the nineteenth century, in particular Neoclassical Theory. However, even though scientific ideas have evolved, mainstream economic theory is still connected to the classical logic of the world. For this reason, economic models are based on rigid assumptions, mathematical formalism and methodological reductionism. It is necessary, therefore, to explore a more coherent approach to economics and finance that includes the perspective of contemporary science, Chaos Theory and the Science of Fractals.

I. NEOCLASSICAL THEORY

1.1 Neoclassical Assumptions

The theoretical background of Neoclassical Theory is found in the classical postulates of Adam Smith (the invisible hand) and the school of Utilitarianism. However, it was properly developed in 1870 by Carl Menger, William Stanley Jevons and Leon Walras. Later, the economist Alfred Marshall will codify neoclassical economic thought in his book “Principles of Economics” published in 1890. Although Neoclassical Theory has become an “umbrella” for other economic postulates, it shares the following core assumptions:

- People have rational and homogeneous preferences, thus they will always choose the outcome with the highest value;
- Individuals act as rational agents by maximizing their utility at any given point in time;
- Agents will act independently based on the relevant available information.

Neoclassical models further assume perfect market competition, no information asymmetry and no transaction costs.

In general, Neoclassical Theory postulates that perfectly informed rational individuals, who just differ with respect to their utility functions, act as self-interested agents trying to optimize their resources. The market, as a result, will try to reconcile these conflicting desires until it reaches a state of equilibrium. In this point, known as Pareto optimum, any change will imply the destabilization of the system. For this reason, this school of thought concludes that markets will allocate scarce resources efficiently via the interaction of demand and supply.

32 Utilitarianism is an economic theory that explains the value of a product in terms of the different utility functions of consumers. Neoclassicals were influenced by their idea that utility can be measured, and because of this, market participants can act as rational individuals.
Based on this idea of market behaviour, Neoclassical theory built a structure to understand the functioning of all the markets in the economy, including financial markets. In fact, both the founder of equilibrium analysis, Leon Walras and the founder of partial equilibrium analysis Alfred Marshall, cited stock exchange as real world cases of competitive markets that clearly exemplify the mechanical process by which market prices are established. Consequently, from this neoclassical perspective, financial theories of competitive equilibrium, such as the Efficient Market Hypothesis and Random Walk Theory, will be developed.

1.2 The Efficient Market Hypothesis and Random Walk Theory

The research on financial markets can be traced to the pioneer work of Louis Bachelier. In his Ph.D dissertation titled “The Theory of Speculation” (1900), Bachelier offered the first statistical method for analyzing the random behaviour of stocks, bonds, and options. The main insight of his investigation was that in a fair market\(^ {33} \), price changes in either direction or a given amount have the same likelihood of occurrence. Consequently, the mathematical expectation of any speculator is zero\(^ {34} \). “Under these conditions, one may allow that the probability of a spread greater than the true price is independent of the arithmetic amount of the price and that the curve of probabilities is symmetrical about the true price” (Bachelier 1900 p 28).

In spite of Bachelier’s remarkable contribution, the interest in the analysis of the market from this point of view developed very slowly. Just with the unprecedented crash of the stock market of 1929, academics began paying attention to his propositions. Afterwards, the study of random markets was mainly conducted to prove that the investment world was highly competitive, and hence prices reflected the best information about the future. Nonetheless, researchers did not go beyond this basic intuition and model the economics involved. To a large extent the empirical work in this area preceded the development of a proper economic discourse. As Fama (1970) explains:

“The impetus for the development of a theory came from the accumulation of evidence in the middle 1950’s and early 1960’s that the behaviour of common stock and other speculative prices could be well approximated by a random walk. Faced with the evidence, economists felt compelled to offer some rationalization. What resulted was a theory of efficient markets stated in terms of random walks, but usually implying some more general ‘fair game’ model” (Fama 1970 p 389).

\(^{33}\) As it is observed, Bachelier assumes a fair market where influences that determine fluctuations are reflected in prices.

\(^{34}\) This idea implies that if an initial investment equals to $100, it would be equally probably that at the end of the period the value moves to $100 + k or $ 100 – k. In other words, the probability of a gain is equal to the probability of loss, and hence the expected value in any future time remains $100 and the expected gain equals to 0.
Consequently, until 1970, Fama developed the theoretical framework for random markets known as the **Efficient Market Hypothesis (EMH)**. This theory states that a market is efficient in the determination of a “fair” price when all available information is instantly processed as soon as it reaches the market, and is immediately reflected in the value of traded assets. Fama stated that for a market to be efficient, it must satisfy the following conditions: “1) there are no transaction costs in trading securities; 2) all available information is costlessly available to all market participants; 3) all agree in the implications of current information for the current price and distributions of each security” (Fama 1970 p 387).

In such frictionless world, investors earn a competitive expected return in the market, as all the cost and benefits associated with a value are already incorporated in the price. According to Fama, the competition between sophisticated investors allows the stock market to consistently price stocks in accordance with the best expectations of the future economic prospects (in Glen 2005 p 92-93). Thus, if the price deviates from its fundamental value, market participants will correct it. At the end, prices will be in a competitive equilibrium with respect to information.

The random walk theory is an extension of this EMH. The **Random Walk Hypothesis** states that random information is the only cause for changes in prices. Therefore, in the absence of new information there is no reason to expect any movement in the price, and the best possible forecast of the price of tomorrow will be, then, the price of today. As a result, the probability that changes occur can be as well determined by a chance game such as tossing a coin to obtain head or tails. This implies that the next move of the speculative price is independent of all past moves or events or, as in mathematics, they form a sequence of identically and independently distributed (i.i.d) random variables.

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35 Fama distinguished three forms of market efficiency. The weak form, which stipulates that current asset prices already incorporate past price and volume information. This means that investors cannot use historical data to predict future prices. For this reason technical analysis are not useful to produce excess returns and thus some fundamental analysis is required. The semi-strong form that argues that all the publicly available information is instantly reflected in a new price of the assets. In this case, not even fundamental analysis will be helpful to produce excess returns. And finally the strong form, which assumes that prices reflect all available information. Hence prices not only take into account historical and public information, but also private information. However, generally the term “all available information” represents the idea that prices reflect all publicly available information.

36 In this point it is important to highlight that Fama does acknowledge that frictionless markets are not met in practice. Therefore, these conditions are sufficient but not necessary. As long as in general prices fully reflect information, there is a “sufficient number” of investors that have ready access to information, and disagreement among investors do not imply an inefficient market, then the EMH can still reflect rational market behavior.

37 In statistics, independence means that “the probability distribution for the price changes during time period $t$ is independent of the sequence of price changes during previous time periods. That is, knowledge of the sequence of price changes leading up to time period $t$ is of no help in assessing the probability distribution for the price change during time period $t$ ”(Fama 1965 p 35). Nevertheless, perfectly independence can not be found in stock markets, thus to account for independence is just necessary that past history of the series would not allow prediction of the future in a way which makes
As mentioned before, this was already formulated in Bachelier’s attempt to mathematically explain a random walk in security prices. Fifty years later, Osborne expanded Bachelier’s ideas and developed in 1959 the Bachelier-Osborne model. This model takes the idea of independence, and further assumes that “transactions are fairly uniformly spread across time, and that the distribution of price changes from transaction to transaction has finite variance” (Fama 1965 p 41). If the number of transactions is large, the distribution of i.i.d random variables will conform to the normal or Gaussian distribution due to the Central Limit Theorem. The normal distribution has the following desirable properties. First, the entire distribution can be characterized by its first two moments: the mean, which represents the location, and the variance, the dispersion. Second, “the sum of jointly normal random variables is itself normally distributed” (Jorion 2007 p 85).

Within this framework, a random walk is represented statistically by a Brownian Motion. A Brownian movement can be defined as a stochastic process that shares three basic properties: homogeneity in time, independence of increments and continuity of paths. Specifically, a Brownian motion is a process such that: 1) there is statistical stationarity, meaning that the process generating price changes stays the same over time. Thus, if \( X_t \) denotes the process at \( t > 0 \), the process \( X_{t_0+t} - X_{t_0} \) has the same joint distributions functions for all \( t > 0 \); 2) the increments of the process for distinct time intervals are mutually independent; and 3) the process declines in a continuous frequency, implying that most price changes are small and extremely few are large, and they change in a continuous manner. This excludes processes with sudden jumps, for instance. Therefore, this model assumes that small movements from \( t_0 \) to \( t \) can be described as:

\[
X_t - X_{t_0} \sim \mathcal{N}(0, (t - t_0)^H)
\]

expected profits greater than they would be under a buy-and-hold model. Fama in his discussion of independence concludes that “the stock market may conform to the independence assumption of the random walk model even though the process generating noise and new information are themselves dependent.” (Fama 1965 p 39).

The normal distribution is also referred as Gaussian, because it was Karl F. Gauss (1877-1955) the one who introduced it when studying the motion of celestial bodies. (Jorion 2007 p 84)

“If price changes from transactions to transaction are independent identically distributed random variables with finite variance and if transactions are fairly uniformly spaced through time, the Central Limit theorem lead us to believe that price changes across differencing intervals such as a day week or month, will be normally distributed since they are the simple sum of the changes from transaction to transaction” (Fama 1963 p 297).

The term “Brownian Motion” comes from the area of physics used to describe the irregular movement of pollen suspended in water, a phenomenon studied by the British physicist Robert Brown in 1928. Again, Bachelier was the first one to suggest that this process could also describe the price variations of financial series.

A stochastic process is determined by a deterministic component and a random variable, which can be only assessed by probability distributions.

For instance, tackling again the game of the coin-toss, statistical stationarity means that the coin itself does not change.
where \( e \) is a standard normal random variable distributed with zero mean and variance proportional to the differencing interval \( T^2 \); and \( H = 0.5 \). In short, in a Brownian Motion, to be able to find \( X_t \), a random number \( e \) (chosen from a Gaussian distribution) is multiplied by the increment \( |t - t_0|^H \), and the result is added to the given position \( X_{t_0} \).

The EMH and Random Walk Theory have led to an enormous source of empirical studies. Some of them have ruled in favor, and others have disproven its theoretical as well as mathematical assumptions\(^{43}\). Nevertheless, it is undeniable that today this theory is the building block for financial pricing models and risk management procedures\(^{44}\). Therefore, Neoclassical Theory is considered the predominant paradigm in economics and finance, and most important, the intellectual bedrock where Wall Street sits.

### 1.3 Failures and Inconsistencies of Neoclassical Theory\(^{45}\)

#### 1.3.1 Discrepancy with economic reality

“Perhaps the biggest failure of the EMH is its consistent failure to mirror or model empirical evidence. Markets have not shown themselves to be efficient and neither has the theory”

(De Bondt 2003 p 201).

While studying the performance of market prices, researchers have discovered certain market behaviors that contradict the idea of efficient markets. For instance, they have found correlation of asset returns with market-book ratios, the firm size or even with the different seasons of the year\(^{46}\). With respect to the former, it has been observed that returns in the month of January generally surpass those of any month. Ikenberry and Lakonishok (1989) observed that particularly small firms performed excessively well relative to large firms in this

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\(^{43}\) For empirical evidence that supports the EMH refer to Fama (1970) and Lo (1997). Research that disproves the EMH is found in Fama (1965), Guimaraes, Kingsman and Taylor (1989), Lo (1997), and Lo and MacKinlay (1999). As it is observed, Fama and Lo are in both sides of the debate, reinforcing the idea that empirical research in the EMH is inconclusive. As Lo and MacKinley said (1999): “What can we conclude about the Efficient Markets Hypothesis? Amazingly, there is still no consensus among financial economists. Despite the many advances in the statistical analysis, databases, and theoretical models surrounding the Efficient Markets Hypothesis, the main effect that the large number of empirical studies have had on this debate is to harden the resolve of the proponents on each side”. (Lo and MacKinley 1999 p 6).

\(^{44}\) Within Modern Portfolio Theory the most important pricing models are: the Capital Asset Pricing Model and the Arbitrage Pricing Theory. Both models assume self-interested rational individuals with homogenous expectations, a frictionless market in equilibrium, and that the distributions of returns is normal. Risk models will be constructed in the same basis. For instance, the standard technique to calculate Value-at-Risk, which measures market risk, assumes returns as normal. Another examples are the Black and Sholes model for option pricing and the Merton Model for default risk, which are based on the idea that a Brownian Motion drives the paths of price returns.

\(^{45}\) From this point onwards, the term neoclassical theory will include the Efficient Market Hypothesis and Random Walk Theory.

\(^{46}\) Again, for research in this topic refer to Guimaraes, Kingsman and Taylor (1989). Lo (1997), and Lo and MacKinlay (1999).
month. Furthermore, these authors discovered the day-of-the-week effect\(^{47}\), and the holyday and turn-of-the-month effect\(^{48}\). Their research concluded with the following statement:

“Seasonal patterns have been identified in an overwhelming number of papers using different time periods, applying various statistical techniques, using equity and non-equity assets not only in the United Stated but also across world markets. The fact that seasonal patterns continue to appear in such diverse data sets provides indication that some form of investor behaviour is the underlying cause. The volume of evidence appears to reject the possibility of spurious fit in the data. Hence, we should take these anomalies seriously and look for explanations” (Ikenberry and Lakonishok 1989 p 106)

Indeed, there is a strong connection between market “anomalies” and the irrational character of market participants. Behavioral researchers have found that individuals have cognition limitations reflected in framing problems\(^{49}\) and heuristics in judgments and learning. These last ones represent biases such as anchoring, overconfidence and illusion of correlation, among others. Furthermore, individuals rely on their emotions when they engage in investment decisions\(^{50}\). Financial bubbles, for example, are partly caused by euphoric and greedy investors who are driven by over optimism and herd behavior (De Bondt 2003). When the market reaches a critical point, volatility rises up, correlation increases and prices experience large and discontinuous decreases.

On an overall basis, there is evidence for season anomalies, for price overreaction, for excess volatility, and for many other situations, which demonstrate that prices are not in their equilibrium values, investors do not act as rational individuals, and markets do not follow a random walk. This does not implies that there is something abnormal with the behaviour of the financial market, but just that neoclassical postulates fail to describe the actual behaviour of financial markets. As Lo (1997) explains: “What are we to make of these anomalies? Their persistence in the face of public scrutiny seems to be a clear violation of the EMH. After all, most of these anomalies can be exploited by relatively simple trading strategies.” (Lo 1997 p xvi).

\(^{47}\) This effect describes the phenomena that stocks tend to be negative on Monday and relatively high on the last trading day of the week. This effect has even inspired the refrain: “Don’t sell stocks on (blue) Monday!

\(^{48}\) Ikenberry and Lakonishk (1989) observed that returns were higher on the last trading day before a holyday and around the turn of the month.

\(^{49}\) Framing problems affect representation, judgment and the selection of behaviour.

\(^{50}\) Warren Buffet, one of the most successful investors in history, explained the behavior of markets as a “fellow named Mr. market (…) a man with incurable emotional problems. At times, he feel euphoric and can see only the favorable factors; while at others he is depressed and can see nothing but trouble ahead for both the business and the world” (in Weiss 1992).
Several authors, such as Mirowski (1990, 2002, 2004), Hsieh and Ye (1998), Chorafas (1994), Peters (1994), Foster (2005), and Faggini and Lux (2009) among others, have highlighted that the discrepancy of economic theory with reality rests in the fact that neoclassical theory resembles Newtonian physics. This thesis will support this argument, showing that it is possible to observe certain key concepts of Newton in Neoclassical economical discourse and method.

1.3.2 Newton’s assumptions in Neoclassical Theory

When Neoclassical Theory hypothesizes that the sum of rational individuals conduces the economy to an optimum equilibrium, it is possible to distinguish the Newtonian idea of modeling a linear system that exhibits a stable and well-defined “equilibrium”. The inclusion of the equilibrium concept into Neoclassical Theory has important metaphorical implications. In Newtonian physics the notion of equilibrium is tied to a body in rest, which can only achieve motion when the system is disturbed with an external force. In this case, the system will change in the same direction and proportional to the force applied. Therefore, by emphasizing equilibrium, Neoclassical Theory has fashioned a model of market behavior in which deviations from “equilibrium” prices are corrected proportionally by the interplay between demand and supply. Furthermore, it has conceived the economy as a system that can only be disrupted by external factors, excluding self-cause spontaneous changes and nonlinear internal motion.

Newtonian rigid determinism and timeless dynamics are also present in Neoclassical Theory (Mirowski 2004). In the Neoclassical realm, markets behave like a mechanical perfect roulette wheel, where no opportunities of arbitrage are permitted. This mechanistic perspective allows the description of market behavior with equations (usually linear) that can connect numerical measurements at a given time to their past and future values. Indeed, when statisticians hypothesized in the Random Walk Hypothesis that the course of a stock price follows a Brownian motion, they do not imply that prices cannot be forecasted. In the contrary, “they merely imply that one cannot forecast the future based on past alone” (Cootner 1964b p 80). For that reason, after the random variable, commonly referred as white noise, has been “separated out”, deterministic equations can actually describe price

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51 Recall that Laws of Motion explained in Part I.
52 Arbitrage refers to the possibility of profiting by exploiting price differences in the market without incurring in additional risk. In an efficient market, the competence between investors would not allow this possibility, therefore, it is said to be an arbitrage-free market.
53 “This is observed in Fama’s work, who concluded in his 1965 research that: “it seems safe to say that this paper has presented strong and voluminous evidence in the random walk hypothesis”, and then, introduced a paper with the sentence, “there is much evidence that stock returns are predictable” (in Jorion and Khoury 1996 p 326).
changes. As Miroswki said: “These stochastic “shocks” had little or no theoretical justification, but themselves seemed only an excuse to maintain the pure deterministic ideal of explanation in the face of massive disconfirming evidence” (Miroswki 2004 p 231).

As a result, this thesis argues that by adopting Newton’s postulates, Neoclassical Theory inherited its limitations of describing the world. In physics Newtonian mechanics are just seen as idealized systems that account for an approximation of reality, and hence can just describe specific cases in nature. In economics, Neoclassical Theory plays the same role as an economic and financial paradigm. It only describes an ideal world with rational individuals in a perfectly reversible and deterministic economic system with no arbitrage opportunities that tends to a stable state of equilibrium.

1.3.3 Newton’s mathematics and method in Neoclassical Theory

Besides Newton concepts, classical mathematics, which involve linear systems with smooth and continuous changes, and symmetric distributions, are part of Neoclassical Theory. In fact, the importance of this type of mathematic language is clearly reflected in the development of the theory. As it was mentioned before, the EMH was created to explain the random character of markets. In particular, it justified the use of statistical tools that required independence and Gaussian distributions. As Peters (1994) said: “the EMH, developed to make the mathematical environment easier, was truly a scientific case of putting the cart before the horse” (Peters 1994 p 41). Accordingly, the Newtonian mathematical idiom and reductionist methodology became essential for the Neoclassical approach to economics.

The most relevant consequence is that since the beginning, Neoclassical theorists refused to explain phenomena that contradicted their assumptions or could not fit in their mathematical equations (Hsieh and Ye 1998). Therefore, large changes in prices or irrational behavior of individuals were just seen as been anomalous. Furthermore, it distanced them from the actual information revealed by real data (Mandelbrot 2004). As a result, neoclassical doctrines are not grounded to empirical observation or even reasonable assumptions. As Mandelbrot (2004) observes: “the fact that mass psychology alone, might have been sufficient evidence to suggest there is something amiss with the standard financial models” (Mandelbrot 2004 p 170).

54 Just recall that a Brownian Motion describes the path of a stock in small independent changes that are distributed with the normal symmetric bell-shape curve.

55 This is argues by Chorafas (1994) and Foster (2005) Citing Foster (2005): “Why should eminently reasonable propositions concerning the existence of time irreversibility, structural change and true uncertainty in historical processes have been so unpalatable to the mainstream of the economics profession? Because of the chosen language of scientific discourse, namely mathematics. A scientific desire to use mathematics as formal medium for deduction. The problem does not lie in the chosen economics but, rather in its limited expression, in the chosen language of discourse”. (Foster 2005 p 371)
Overall, the failure of Neoclassical Theory is its chosen mode of discourse and set of tools. It took the ideas of the mid-19th century prior to the Second Law of thermodynamics, and remained mired in its original orientation even though the economic world has changed enormously (Mirowski 2002 and 2004). Therefore, this thesis proposes that to capture the complexity of the global economy and financial markets, it is necessary to renew our economic and financial theories with the perspective of Chaos Theory and the Science of Fractals. This would allow them to address the market with a different vocabulary, and more important, with a different method.
II. APPLYING CHAOS THEORY AND THE SCIENCE OF FRACTALS TO ECONOMICS AND FINANCE

“The argument for success no longer makes a good case for the mechanistic view of the world” (Lavoie 1989 p 613)

2.1 Complexity Economics

The economic discourse of Chaos Theory can be found in complexity economics. This new school of thought emerged with the necessity of economists to renew the Newtonian vision and allow a more coherent perspective of economic reality. In Rosser (2004) words:

“Increasingly in economics what had been considered to be unusual has [now] come to be considered usual and acceptable, if not necessarily desirable. Whereas it had been widely believed that economic reality could be reasonable described in sets of pairs of linear supply and demand curves intersecting in single equilibrium points to which markets easily and automatically moved, now it is understood that many markets and situations do not behave so well. Economic reality is rife with nonlinearity, discontinuity, and a variety of phenomena that are not easily predicted or understood. The order of the economy appears to emerge from the complex interactions that constitute the evolutionary process of the economy. Even what seems simple in economics generally arises from behaviour not reflecting rational expectations; we live in a world that reflects the enormous variety and diversity in their knowledge, attitudes and behaviour interacting with each other in an enormous range of institutional frameworks” (Rosser 2004 p ix)

Consequently, contrary to Neoclassical assumptions, complex economics describe the economy as a non-equilibrium system with highly nonlinear behaviour composed by heterogeneous individuals with complex interdependencies. As in Chaos Theory, within this new framework, systems are characterized by having emergent properties that appear from unexpected behaviour. At some point in time, the system can reach stability, yet it is easily broken as suddenly changes occur. Thus, the system will never have a single point of equilibrium, but instead multiple equilibria or even work in far away from equilibrium.

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56 It is hard to establish a certain point in time where the rupture with the Newtonian paradigm was made in economics. However texts that include concepts of chaos theory in economics can data since approximately 1960.

57 This definition is taken from Arthur, Durlauf and Lane (1997) which characterize complex economic systems as having: “1) Dispersed interaction among heterogeneous agents acting locally on each other in some space, 2) No global controller that can exploit all opportunities or interactions in the economy even though there might be some global interactions, 3) Cross-cutting hierarchical organization with many tangled interactions, 4) Continual adaptation by learning and evolving agents, 5) Perpetual novelty as new markets, technologies, behaviors, and institutions create new niches in the ecology of the system, and 6) Out of equilibrium dynamics with either zero or many equilibria existing, and the system unlikely to be near a global optimum” (in Rosser 2004 p x).
dynamics. Chaotic economic motion is reflected in the continuous fluctuations between valleys, peaks, and bumps of economic cycles. In this process, the system experiences periods of stability, critical points, and then self-organization dynamics (Hayek 1967 and Prigogine 1984).

Complexity authors have used the “sand pile model” to explain the cyclic behaviour of an economy system such as the financial market. In this case, asset prices grow exponentially due to some exogenous or endogenous change in the system. This could be a technological change or a new pattern of behaviour, for instance. The speculative bubble will grow until the market reaches a critical point that inevitably leads to the burst of prices. As in the avalanches in the “sand pile” model, the distribution of these events exhibit greater variance than the random “drops of grains” that caused the “pile to grow”. Afterwards, the financial system self-organizes producing changes in economic activities and allocation of resources (Brock 1991).

Under these chaotic conditions, it is not possible for agents to behave optimally and to adequate information in order to form rational expectations, like Neoclassicism assumes. Consequently, this theory conceptualizes market participants as having limited knowledge and imperfect foresight of future outcomes. Furthermore, agents are characterized as continuously changing and learning from the evolving environment. Therefore, one of the key features of this new paradigm is that the economical model is a system where expectations affect actual dynamics and actual dynamics feedback into the expectations scheme. Thus, agents with their natural cognition limitation will try to learn, adapt and update their parameters in accordance with observed behaviour.

As it is observed, complex economics contradicts two main neoclassical postulates. First, that the economy is a close linear system that tends to a state of equilibrium; and secondly, that market participants behave as rational individuals and have rational expectations. Instead, it proposes more coherent assumptions about the behaviour of the economical system and individuals (see table 2.1). This has allowed economists to understand better

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58 “Keynes was the original supporter for unstable dynamics, suggesting that it would be coincidental to find macroeconomic equilibrium at any point in time due to the complex interrelationships and movements of so many variables and the large role that psychology plays in the market” (in Carrier 1993 p 300)
59 This literature has its theoretical roots in the early nonlinear business cycles of Goodwin (1982).
60 Contrary to Neoclassical assumptions, complexity economics emphasizes that “even in the absence of external shocks, learning by itself might generate convergence to complex nonlinear dynamics”. (Grandmont 1998 p 163)
61 This argument has been even accepted by one of the early leading defenders of rational expectation, Thomas Sargent (Rosser 1999 p 183)
62 Indeed, these new assumptions concur with the ideas of behavioral economics, evolutionary economics, Institutional Theory and the Austrian School.
market dynamics. Furthermore, it has originated a shift in their method, opening them the opportunity to explore the complexity of economic phenomena. Again, citing Rosser (2004):

“This change of perspective has brought forth new approaches to analysis. Previously, a premium was placed on deductive formal proofs of theorems that sought to derive general solutions broadly applicable, a view that reached its culmination in the French Bourbakist school\(^3\). Now we see a greater emphasis on computer simulations and experimental methods to inductively determine possible outcomes and ranges of solutions. Emergent phenomena from complex systems are not usually discovered by theorems, but more frequently by the use of increasingly powerful computers to explore limits and possibilities that can arise. Awareness of the ubiquity is transforming the way that we think about economics” (Rosser 1999 p ix)

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<th>TABLE 2.1: COMPARISON BETWEEN NEOCLASSICAL AND COMPLEX ECONOMICS</th>
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<tr>
<td><strong>NEOCLASSICAL ECONOMICS</strong></td>
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<tr>
<td>Origins from Physical Sciences</td>
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<tr>
<td>Systems</td>
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Conclusively, complex economics left behind the neoclassical uniqueness of equilibrium, assumptions of rationality, and overbearing determinism, to develop a more coherent explanation of economic systems. A continuation, it will be demonstrated how Chaos Theory and the Science of Fractals can contribute to the development of financial theory.

\(^3\) Group of mathematicians (mainly French) who wrote a series of books in 1935 presenting their knowledge in advance mathematics. Their postulates are characterized by its rigour and generality.
2.2 Chaos Theory and the Science of Fractals in Finance

Both in physics and finance, the objective of Chaos Theory and the Science of Fractals is to study the aperiodic non-linear behaviour emerging from systems sensitive to the initial conditions that tend to follow paths characterized by “strange attractors”. Accordingly, the disordered behavior is a local property of the system, but there are in fact some distinguishable patterns of market behaviour. This is the main insight that this new paradigm gives to finance. It describes markets as having local randomness and global determinism, just as in fractal structures on nature. Therefore, this thesis will argue that by adopting a fractal perspective to the market, it would be possible to understand better market dynamics.

2.2.1 From the Efficient Market Hypothesis to The Fractal Market Hypothesis

Peters (1994) proposed the Fractal Market Hypothesis (FMH) as a new framework for modeling the conflicting randomness and deterministic characteristic of capital markets. For the moment, the FMH seems to be a robust theoretical contribution for understanding the turbulence, discontinuity, and non-periodicity that truly characterize today’s capital markets.

The Fractal Market Hypothesis explains that a financial market is a place where investors meet to find a buyer or a seller at any point in time. However, to allow investors with different time horizons to trade efficiently and at a price close to its fair value, it is necessary to ensure sufficient liquidity in the market. In fact, a liquid market is a guarantee that there will not be any panic or crash if supply and demand become imbalanced. Therefore, the main function of markets is to provide a stable, meaning a liquid environment, for trading activity.

Markets will find this stability in “investors with different horizons, different information sets, and consequently, different concepts of ‘fair price’” (Peters 1994 p 43). According to Peters (1994), individuals with diverse time horizons will value information differently. For instance, due to the fact that day traders are concern with daily prices, they will pay more attention to recent trends, and as a result, will disregard information about the future. On the other hand, long-term investors are more likely to have as priority long-term prospects. For this reason, investors will have a different concept of what a fair price is, and consequently, information circulating into the market will have a different impact on each investment horizon. In this way, if some new piece of information causes a drop in the price on the short-term horizon, then long-term investors can keep the stability of the market by buying the stocks. Again, they do so because they do not value the information as highly and can absorb this short

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64 As Haridas (2003) explains, “the shorter the investment horizon, the greater the importance of technical factors; conversely, the longer the investment horizon, the greater the importance of fundamentals” (Haridas 2003 p 24).
The main assumption here is that investors share the same risk levels (once and adjustment is made for the scale of investment horizons). Indeed, this “shared risk explains why the frequency distribution of returns looks the same at different investment horizons” (Peters 1994 p 46), and mainly, is the reason for the self-similar characteristic of capital markets. If the market loses this “fractal” structure it will become unstable.

The collapse of markets may occur when long-term expectations become uncertain. For example, political crisis, wars, natural disasters can change the fundamentals of the market. As a result, long-term investors, which are directly affected by these changes, will become short or stay out of the market. If positions are shortened, liquidity will be scarce, and consequently, the market will enter into a critical period. Peters (1994) further explains:

“As long as investors with different investment horizons are participating, a panic at one horizon can be absorbed by the other investment horizons as a buying (or selling) opportunity. However, if the entire market has the same investment horizon, then the market becomes unstable. The lack of liquidity turns into panic. When the investment horizon becomes uniform, the market goes into ‘free fall’; that is, discontinuities appear in the pricing sequence. In a Gaussian environment, a large change is the sum of many small changes. However, during panics and stampedes, the market often skips over prices. The discontinuities cause large changes, and fat tails appear in the frequency distribution of returns. Again, these discontinuities are the result of a lack of liquidity caused by the appearance of a uniform investment horizon for market participants”. (Peters 1994 p 47)

As it is observed, this proposition is consistent with what is known by neoclassical theory as “irrational” market behavior. More specifically, it recognizes that investors may overreact to events and information, and this can cause large price movements or fat tails in the distribution. In addition, herding behavior is seen as the point where the entire market trades on the same information set (technical analysis).

Lastly, the FMH argues that capital markets will have a “short-term fractal statistical structure superimposed over a long-term economic cycle, which may be deterministic” (Peters 1994 p 48). This theory views economic cycles as possible strange attractors of the economic system. In other words, if the economic activity is characterized by economic growth, the market will approach that growing tendency. Economic cycles will, therefore, dominate the financial market. “Since long term-economic activity is less volatile, this will make long-term returns less volatile as well. The variance of returns would then be bounded by economic activity” (Haridas, 2003 p 24). However, if a security is not tied to the economic cycle, then
there will be no long-term trend, and trading activity will continue to be important, even at long horizons.

Recapitulating, the FMH proposes the following:

1. The market is stable when is made up of a large number of individuals with different investment horizons.
2. The impact of information depends on the different investment horizons. Therefore, price changes are a reflection of information important just to that investment horizon.
3. If an event changes the time horizon of investors to a uniform level, the market will become unstable.
5. The “strange attractors” in capital markets are economic cycles. In case a security is not tied to the economic cycle, there will not exist a long-term trend. As a result, trading activity, liquidity and short-term information will predominate.

Comparison between the EMH and FMH

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<th></th>
<th>Efficient Market Hypothesis</th>
<th>Fractal Market Hypothesis</th>
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<tbody>
<tr>
<td><strong>Focus</strong></td>
<td>Efficiency of markets and the fair price of assets</td>
<td>Liquidity</td>
</tr>
<tr>
<td><strong>Market</strong></td>
<td>Market is in equilibrium</td>
<td>The market cannot reach just one equilibrium as each investment horizon has different equilibria states.</td>
</tr>
<tr>
<td><strong>Validity of Normal Distribution</strong></td>
<td>Variables are normally distributed.</td>
<td>General distributions have high peaks and fat tails. These are caused by discontinuous price jumps in the market. Moreover, because large changes are few, the variance is infinite.</td>
</tr>
<tr>
<td><strong>Market memory and cycles</strong></td>
<td>Markets behave randomly thus past events have no effects. The best guess of the possible price tomorrow is the price of today plus a random term.</td>
<td>History plays an important role in determining the path of the system. In fact, the market exhibits a local fractal structure and a global deterministic order tied to economic cycles. Prediction is then possible just in the short term.</td>
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</table>
As it is observed, the FMH highlights the impact of liquidity and different investments horizons on the financial behavior of investors. Different from the EMH, this heterogeneity is central in the FMH. Furthermore, it recognizes that markets do not tend to equilibrium, investors are not rational, and most important, financial systems are not deterministic. In fact, this new hypothesis emphasizes its fractal structure. Therefore, with this new perspective, it is possible to have a more coherent understanding of how financial markets work. Most important, it provides the theoretical framework for the development of new statistical tools for analyzing fractal markets.

2.2.1 From a random walk to a fractal approach to the market

A shift from an “efficient” market to a “fractal” market has further implications. Financial systems that combine local randomness as well as global determinism cannot be explained by a random walk. Therefore in order to study these systems, it is necessary to find a new statistical description of capital markets. This will signify a change from Gaussian statistics to Fractal statistics. In short, Chaos theory and the Science of Fractals will not only change the theoretical assumptions, but also the mathematical language and method in finance.

a) A Fractal Financial Market

Benoit Mandelbrot, father of fractal geometry, indeed first discovered the distinctive characteristics of fractals in financial time series. This author approached the market as a scientist, not a deductive mathematician, and in doing so, he observed that the variance of prices misbehaved leading to abnormally big changes. This behaviour was reflected in high peak and “fat” tailed distributions, which frequently followed a power of law. However, the most peculiar property was that these leptokurtic distributions appeared unchanged whether the time scale was weekly, monthly or yearly. Therefore, Mandelbrot concluded, “the very heart of finance is a fractal” (Mandelbrot 2004 p 165).

The importance of Mandelbrot’s discovery is that it highlights that under the apparent disorder of capital markets, there are some “stylized facts” that can describe the behaviour of capital markets. Borland, Bouchard, Muzy and Zumbach (2005) characterized them as universal, in the sense that are common across different assets, markets and epochs. Similar to Mandelbrot insights, these authors found that empirical data is characterized by certain

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65 Later, as many economist refused to accept his ideas, he started losing interest in financial fractals, and hence turned to physics. In this field, he would be able to develop the fractal geometry of nature.
66 This implies that graphs will not fall toward zero as sharply as a Gaussian curve.
67 Leptokurtis is the condition probability that distributions have fatter tails and higher peaks than the normal distribution distributions.
68 The movements of stocks, bonds or currencies, undoubtedly, all look alike when a market chart is reduced or enlarged.
qualitative properties: 1) the distribution of returns is in fact non-Gaussian, especially for short intervals of time that have a stronger kurtosis\(^{69}\); 2) volatility is intermittent and correlated what is known as volatility clustering; 3) Price changes scale anomalously with time (“multifractal scaling”\(^{70}\)). These are, indeed, not statistical irregularities, but the rules of market behavior.

Evidently, under these new assumptions of market behavior, it is not possible to represent variation of prices by the neoclassical random walk. Therefore, Chaos Theory and the Science of Fractals propose alternative tools that can account for discontinuities, patterns and dependence. This will imply a change in the assumptions of normal distributions and Brownian Motions.

**b) From Normal distributions to Stable Paretian Distributions**

“Fractal statistics” incorporate the idea that large and discontinuous price changes are far more common than what the Gaussian hypothesis predicts. As observed in financial markets, transactions occur at different instants of time and are quoted in distinct units; hence, mathematically speaking a price series is never continuous. If prices move smoothly from one value to the other, it is possible to approximate the distribution to a normal. But prices are merely discontinuous as they go up or down very steeply, and even more, they tend to group. Thus, within this framework, discontinuity is in fact a very common property of financial markets, and it is reflected in the “fat tails” of the distribution\(^{71}\).

Mandelbrot (1997) explains discontinuity with the famous biblical story of the deluge, naming it the **Noah effect**. As the Old Testament narrates, God ordered the Great Flood to purify the world. However, he first put Noah in charge of building a strong ship to save his family and a representation of the earth’s species. For this author, the Noah story resembles the large and discontinuous changes that appear in capital markets. In Mandelbrot (2004) words:

“The flood came and went – catastrophic, but transient. Market crashes are like that. The 29.2 percent collapse of October 19, 1987, arrived without warning or convincing reason; and at that time, it seemed like the end of the financial world. Smaller squalls strike more often with more localized effect. In fact, a hierarchy of turbulence, a pattern that scales up and down with time, governs this bad...”

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\(^{69}\)“Kurtosis describes the degree of flatness of a distribution” (Jorion 2007 p 87)

\(^{70}\)As multifractals in nature, multifractal scaling means that prices scale in different size and in different direction.

\(^{71}\)In Mandelbrot’s words: “Discontinuity far from being an anomalous best ignored, is an essential ingredient of markets that helps set finance apart from the natural science (...) [The only reason for assuming continuity] is that you can open the well-stocked mathematical toolkit of continuous functions and differential equations” (Mandelbrot 2004 p 86).
financial weather. At times, even a great bank or brokerage house can seem like a little boat in a big storm” (Mandelbrot 2004 p 200).

To account for discontinuity Mandelbrot proposed in his early work of 1960’s to replace the Normal or Gaussian distribution assumption for the Stable Paretian Hypothesis. Mainly, stable Paretian distributions allow scaling properties or power law relationships, which can account for fat tails and high peaks in the distributions of financial returns.

Scaling distributions were first studied by Vilfredo Pareto, Italian economist, in his investigation of personal income in Italy. Pareto found that social classes were represented with a “social arrow” (not a pyramid), very fat at the bottom representing the poor mass of men, and very thin at the top describing the wealthy elite (Mandelbrot 2004 p 153-154). Pareto then modeled the wealth of individuals using the distribution \( y = x^{-v} \), where \( y \) is the number of people having income \( x \) or greater than \( x \), and \( v \) is an exponent that Pareto estimated to be approximately 1.5. When calculating this relationship to other geographical areas, Pareto found that this result was also applicable to countries such as Ireland, Germany and even Peru.

Pareto’s basic observation of a power of law was very insightful. In his distribution of personal income Pareto involved tails that were heavy and followed a power-law distribution represented by \( \Pr(U > u) = u^{-\alpha} \). In this case, the probability of finding a value of \( U \) that exceeds \( u \) depends on \( \alpha \). Such power laws are very common in physics and are a form of fractal scaling.\(^{73}\)

\(^{72}\) As applied to a positive random variable, the term scaling signifies scaling under conditioning. To condition a random variable \( U \) specified by the tail distribution \( P(u) = \Pr(U > u) \). The power law is due to the characteristic exponent \( \alpha \).

\(^{73}\) Fractals also scale also by a power law, more specifically, in fractals the range increases according to a power.
The long tailed distribution found in Pareto’s work, led the French mathematician Levy, to formulate a generalized density function named Stable Paretian distribution, in which the normal and the Cauchy conform a special case. These distributions can be described by four parameters: $\alpha$, $\beta$, $\delta$ and $\gamma$. The locational parameter is $\delta$, and if $\alpha$ is greater than one, $\delta$ will be equal to the expectation or mean of the distribution. The scale parameter is $\gamma$, and it can be compared to the measure of dispersion. Its value can be any positive number (Fama 1963 pp 298 - 299). When $\gamma = 1$ and $\delta = 0$, the distribution is said to be in its reduced form. Nevertheless, $\alpha$ and $\beta$ are the two parameters that determine the shape of the distribution. The parameter $\beta$ is an index of skewness and must be in the interval between $-1 \leq \beta \leq 1$. When $\beta = 0$ the distribution is symmetric; if $\beta > 0$ the distribution is skewed right, and if $\beta < 0$ is skewed toward the left. On the other hand, $\alpha$ is the variable that describes the total probability contained in the extreme tails. It is called the index of stability or characteristic exponent, and must be in the range from $0 < \alpha \leq 2$. When $\alpha = 2$, the distribution is normal and the variance exists. When $\alpha < 2$, there are more observations in the tails that under the normal distribution, and even more the variance becomes infinite or undefined.

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74 A Cauchy curve is a probability distribution with undefined mean, variance and other higher moments. Although it looks similar to a normal distribution, because of its symmetry and bell-shape curve, it has much heavier tails and higher peaks.

75 “Skewness describes departures of symmetry” (Jorion 2007 p 86)

76 In other words, a small value of $\alpha$, implies thicker tails of the distribution.

77 Infinite variance implies that the variance of the time series changes over different samples and generally increases with the sample size. Therefore, it does not settle down to some constant value as it is assumed with normal distributions. This will entail that the sample variance is not statistically significant. If $\alpha \leq 1$, the same will happen for the mean as it would not exist in the limit. Nevertheless, it is important to point out that as in all fractal structures, there is eventually a time frame where fractal...
Stable Paretian distributions have some desirable characteristics that allow them to describe the patterns of financial markets. First, they are invariant under addition meaning the sum of two Paretian distributions is itself stable Paretian. This stability holds even when the values of the location and scale parameters, $\delta$ and $\gamma$, are not the same for each individual variable. As Fama (1963) explained, “the sum of stable Paretian variables, where each variable has the same value of $\alpha$ and $\beta$ but different location and scale parameters, is also stable Paretian with the same values of $\alpha$ and $\beta$” (Fama 1963 p 301). In fact, because $\alpha$ and $\beta$ are not dependent on scale, although $\delta$ and $\gamma$ are, stable Paretian distributions are consider self-similar distributions. Second, they allow an asymmetric representation of the distribution of returns with high peaks and fat tails. As a result, with these distributions it is possible to model abrupt and discontinuous changes. Examples of this behaviour are found in market critical dynamics that amplify the bullish or bearish sentiment, or when the fractal structure of the market is lost and all investors begin trading in the same investment horizon and with the same set of information, as the FMH explains.

These distinctive characteristics lead Mandelbrot (1963a) to propose the Stable Paretian Hypothesis arguing that “1) the variance of the empirical distribution behave as if they were infinite; 2) the empirical distributions conform best to the non-Gaussian member of a family of limiting distributions called stable Paretian”. (Fama 1963 p 298). His basic idea is to model the percentage changes in a price as random variables with mean zero, but with an infinite standard deviation. In other words, the distribution of speculative prices is defined by the interval $1 < \alpha < 2$, contrary to the Gaussian hypothesis that states that $\alpha = 2$.

For the moment, Mandelbrot’s hypothesis cannot be taken as definite. As it is observed, this statement was proposed in the early 1960’s, and at that time, it received support for a small group of economists, including Eugene Fama. However, with the release of the famous survey of Fama in 1970 about efficient markets, the academy disregarded Mandelbrot’s idea in favor of the Gaussian assumption. Therefore, empirical evidence is not so extensive in this topic, and it is still necessary to prove if indeed $\alpha$ ranges in the interval $1 < \alpha < 2$. It could take lower or higher values, for instance.

scaling ceases to apply, and thus there could be a sample size where variance does indeed become finite. But within our life time (at least 100 years), stable distributions will behave as if they have infinite variance.

Indeed, this is the reason why they received the name stable. Stable means that the basic properties of an object remain unaltered even though it is rotated, shrunk, or even add it to something else.

The empirical analysis of testing the Stable Paretian Hypothesis will not be included in this thesis. Even so, it is important to highlight that some researchers have found that indeed $\alpha$ deviates from the Gaussian case of 2. One of the most recent studies is Johnson, Jeffries and Ming Hui (2003), which investigated the composite index recorded in a daily basis between 1966 and 2000, and the Shanghai stock exchange index recorded at 10-s intervals during the period of eight months in 2002. In this
However, the idea that distributions of returns are in fact non-Gaussian deserves more attention. This proposition is important because it highlights that there are more large abrupt changes, and hence, markets are inherently more risky than those describe by a Gaussian market. Therefore, for the financial industry, accepting Mandelbrot’s hypothesis implies that current models based on the normal distribution are misleading, as they do not account for the real risk in financial markets. In fact, this will mean a change in the assumptions behind the Capital Asset Pricing Model, Arbitrage Portfolio Theory, Black and Sholes models for pricing options, the Merton model for pricing credit risk, and Value-at-Risk. Almost all of the models that Wall Street uses to make financial decisions nowadays would need to be seriously revaluated. As Cootner, MIT economist said in his review of the Mandelbrot (1963b): “Mandelbrot, like Prime Minister Churchill before him, promises us not utopia but blood, sweat, toil and tears” (Cootner1964d p 337). Sable Paretian distributions, however, seem to be a robust assumption as they account for asymmetry, and most important, the "inconvenient" outliers.

c) From a Brownian motion to multifractal analysis

In Chaos Theory and the Science of Fractals, dependence is also an important property for financial time series. In particular, long-term memory demonstrates that aleatory influences in the starting conditions play an important role in shaping the behaviour of a dynamical system in the future. Therefore, contrary to the independence assumption of a random market, in a fractal market past events cannot be excluded. Again, Mandelbrot (1997) explains this behaviour with the biblical story of the seven fat and seven lean years, calling it the Joseph effect.

In this story, the Pharaoh of Egypt dreamed that seven lean cows that rose out of the Nile river devoured seven fat cows. Then, he dreamed again of seven scraggly ears of grain that consumed seven plump ears of corn. Joseph, a Hebrew slave, interpreted his dreams as having seven years of prosperity followed by seven years of famine. Subsequently, he advised Pharaoh to storage some grains, in case bad times where to come. When the seven bad years started, Pharaoh and Joseph opened the storehouses and were able to sell the grain for the starving Egyptians. The Joseph effect is evident whenever cycles are distinguished in economic time series. In Joseph’s story, the cause of the cycle is the Nile River, which can have up volumes for long periods of time, and down levels for others. Because crops in Egypt depend on this river, they will also experience this “long run dependence”.

study, they found that alpha equals to 1.44 for both markets. Although this value can change from market to market, it does prove that markets have scaling properties.

80 To illustrate better this point Part III will show how measuring Value-at-Risk in stable Paretian markets can lead to a significant improve in risk assessment.
In 1997, Mandelbrot began developing financial models that could account for both discontinuities in prices, the Noah Effect, and price dependence, the Joseph effect. As Mandelbrot (2003) explains:

“I came to believe in the 1950s that the power law distributions and the associate infinite moments are key elements that distinguish economics from classical physics. This distinction grew by being extended from independent to highly dependent random variables. In 1997, it became ready to be phrased in terms of randomness and variability falling in one of several distinct “states”. The “mild” state prevails for classical errors of observation and for sequences of near-Gaussian and near-dependent quantities. To the contrary, phenomena that present deep inequality necessarily belong to the “wild” state of randomness” (Mandelbrot 2003 p 6)

The idea of Mandelbrot was, therefore, to develop scaling models that could allow prices to scale up and down, and to the left and to the right. This will mean a generalization of neoclassical models, which only describe “mild” level of randomness, in order to account for “wild” variations in prices. According to Mandelbrot (2003), these models resemble reality more closely, they are parsimonious, and they are creative, in the sense that they can resemble statistically the behaviour of the market based on few assumptions (Mandelbrot 2003 p 6).

Nonetheless, it is important to highlight that fractal models do not purport to predict the future. With Neoclassical models, it is possible to make long-term prediction of the financial systems; whereas with fractal models only short-term predictions are possible. In spite of this, their relevance resides in the fact that they do give a more realistic picture of market dynamics, and hence, a better comprehension of financial risks.

**The Fractional Brownian Motion and Multifractal Analysis**

Consequently, Mandelbrot (1997) proposed the *Fractional Brownian Motion (FBM)*, sometimes referred to as 1/f (fractional) noise. This stochastic process starts with the familiar Brownian motion: the distance traveled is proportional to the same power of the time elapsed. However, in a fractional Brownian motion $H$ can range from zero to one allowing

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81 Mandelbrot introduced this term to represents observations that fluctuate around a “normal state” that represents equilibrium.

82 Describes non-averaging change. Wild processes contain the following characteristics: 1) repeated instances of shared discontinuity; 2) “concentration can automatically and unavailable replace evenness” (Mandelbrot 1997 p 29); 3) non-periodic cycles can follow from long-range statistical dependence.

83 For a random fractal with a prescribed Hurst exponent, it is only necessary to set the initial scaling factor for the random offsets to $\frac{1}{2} \left( |5 + 1|^{2H} - 2|5|^{2H} - |1|^{2H} \right)$. 
the process of price variations to describe the “wild” randomness of financial data. Because of the different fractional values that H can take, it is called a Fractional Brownian Motion. An H=0.5 describes a Gaussian random process, an H < 0.5 means an anti-persistent behaviour\textsuperscript{85}, and an H > 0.5 is related to a persistent case\textsuperscript{86}. An H closer to one indicates a high risk of large and abrupt changes.

The following “cartoons”\textsuperscript{87} illustrate better the difference between a Brownian Motion and a Fractional Brownian Motion with different Hurst Exponents:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fbm.png}
\caption{FBM with different Hurst exponents. The upper graph has an H=0.3, the middle graph has an H=0.5 and the bottom graph has an H=0.75 (Taken from Mandelbrot 2005 p 187).}
\end{figure}

\textsuperscript{84} Here, H is referred to the Hurst Exponent. For a deeper understanding please refer to Appendix I.
\textsuperscript{85} In this case, a positive trend is follow by a negative one, and a negative by a positive. For that reason, an antipersistent system covers less distance than a random process.
\textsuperscript{86} Long-term memory effects characterize a persistent behavior. In other words, what happens today influences subsequent changes or in chaotic terms, initial conditions affect future dynamics. This effect occurs regardless of time scale. For instance, daily returns are correlated with future daily price changes; as well as weekly changes are correlated with future weekly changes. For that reason, it is said that there is no characteristic time scale.
\textsuperscript{87} Mandelbrot named his graphs “cartoons” to avoid misunderstanding with what is known as models. He uses this term “in the sense of the Renaissance fresco painters and tapestry designer: a preliminary sketch in which the artist tries out a few ideas, and which if successful becomes a pattern for the full oeuvre to come” (Mandelbrot 2004 p 117).
Within this framework, two kinds of processes can be distinguished. A \textit{uniscaling} or \textit{unifractal} process, where its scaling behaviour is determined from a unique constant $H$. This is indeed the case of linear self-affine processes and, $H$ is the self-affinity index or scaling exponent of the process. The other is a \textit{multiscaling} process or \textit{multifractal}, where different exponents characterize the scaling of different moments of the distribution. More precisely, it consists in letting the exponent $H$ to depend on $t$, and to be chosen among infinity of possible different values. The key here is to introduce trading time\textsuperscript{88}, as price variations are best not followed in physical clock time but rather in trading time. “To implement this idea in a scaling world, one must identify price variations as a scaling function of trading time, and trading function as a scaling function of clock time”\textsuperscript{89} (Mandelbrot, 1997 p 55).

Mandelbrot (2004) explains multifractal scaling in a more comprehensive manner with the \textit{baby theorem}: “The family starts with the parents. The father takes clock time and transforms into trading time. The mother takes clock time and changes it into a price. Merged together, the baby takes the father trading time and converts it into a price by the rules the mother provides (see figure 2.3)” (Mandelbrot 2004 p 213).

Using Monte Carlo simulation\textsuperscript{90}, Mandelbrot was able to test the model in the computer. The result was a statistical similarity in the behaviour of market prices. It was not completely identical, since the inputs were reduced to a smaller number of parameters, and thus, the outcome was undoubtedly affected. But as Mandelbrot explains:

\textsuperscript{88} Trading time is well defined in the stock exchange as the time that elapses during the open market hours. The introduction of this concept into financial models changes the Newtonian belief of absolute time to the relative concept of Einstein.

\textsuperscript{89} For a more detailed explanation please refer to Appendix II.

\textsuperscript{90} Broadly speaking, Monte Carlo simulations are numerical simulations of random variables made by advance computer techniques. They were first “developed as a technique of statistical sampling to find solutions to integration problems” (Jorion 2007 p 308). The importance of this method is that is an open-form technique, in the sense that it generates a whole distribution of possible outcomes, each of which allows the variables to migrate within predefined limits.
“In financial modeling all we need is a model “good enough” to make financial decisions. If you can distill the essence of GE’s stock behaviour over the past twenty years, then you can apply it to financial engineering. You can estimate the risk of holding the stock to buy your portfolio. You can calculate the proper value of options you want to trade on the stock. This is, of course, exactly the aim of all financial theory, conventional or not. The one difference: This time around, it would be nice to have an accurate model”. (Mandelbrot 2004 p 221).

As it is observed, the importance of these models is that they take into account the “stylized facts” of financial markets, or in mathematical terms the “invariances”, to statistically describe the real behaviour market dynamics.

For a financial theory this will mean a great contribution for the improvement of financial models, as it would generalize Gaussian assumptions to account for volatility clustering and path dependence. Thus, it allows having a better understanding of how risk drives markets, and possible outcomes in the short-term. Therefore, Mandelbrot’s legacy to finance theory is that he proposes new assumptions, to develop models that are based on the observed behaviour of capital markets. After all, no matter how sophisticated is the model for financial analysis; it will be limited by the accuracy and reliability of the underlying assumptions describing the market dynamics.

Nevertheless, it is important to highlight that despite Mandelbrot’s remarkable proposition, the search of a faithful financial model is not over yet. With Mandelbrot’s investigations now it is possible to know that price changes behave very different from a random walk. But being such an undeveloped field, it is still subject to possible improvements⁹¹.

⁹¹ For instance, currently theorists are connecting multifractal Brownian motion with levy distributions to capture jumps, heavy tails, and skewness. This research was motivated since FBM are generated from Gaussian random variables, thus they have less power to capture heavy tails. They are also including GARCH processes and other techniques for improving forecast volatility. For more information of these developments please refer to Sun, Rachev and Fabozzi (2008).
PART II. IMPLICATIONS FOR RISK MANAGEMENT

Nowadays, financial institutions face an increasingly complex economic environment. Therefore, risk management has become an essential tool for the survival of any business activity. Financial Institutions, in particular, have adopted risk management practices as both a shield and a sword to deal with the turbulent capital markets. Indeed, Walter Wriston, former chairman of Citicorp, argued that “bankers are in the business of managing risk. Pure and simple, that is the business of banking” (Jorion and Khoury 1996 p2). For this reason, they must ensure that risks are properly identified, measured, managed and controlled. Chaos theory and the Science of Fractals, hence, present an opportunity to professionals in this area to improve their quantitative and qualitative tools.

I. TOWARDS BETTER QUANTITATIVE MODELS IN RISK MANAGEMENT

1.1. Implications of a fractal market for risk management

As it was demonstrated in the last section, by adopting Chaos Theory and the Science of Fractals in finance the traditional assumptions of how capital markets behave are modified. This new paradigm turns upside down neoclassical ideas, to give a more adequate vocabulary and method to describe capital markets. For risk management, this would mean a change in the way risk is perceived and how it is controlled.

Chaos Theory and the Science of Fractals characterize financial markets as systems sensitive to initial conditions that progress in a non-linear behaviour due to feedback mechanisms. In addition, it conceptualizes agents as having limited cognition capabilities, and most important, behaving irrationally in the market. For risk management this is important because it describes markets not as efficient and stable, but turbulent and volatile. Essentially, it recognizes the risky nature of financial markets.

However, today’s methods to control and price risk are still based on the neoclassical assumptions of normal distributions and Brownian motions. This is probably one of the reasons that explains the failure of risk management systems in times of crisis. In these
models, price changes are assumed to move in a smooth and in continuous manner from one value to another, and extreme events are just considered far outliers improbable to happen.

On the contrary, the objective of fractal tools is to measure the “roughness” of the real dynamics of prices. For instance, stable Paretian distributions account for high peaks and fat tails, and multifractals describe volatility clustering, discontinuity and patterns in their models. As Mandelbrot (2004) explains:

“Suddenly, turbulence ceases to be a metaphor. Multifractals make turbulence a fundamentally new way of analyzing finance. Markets no longer appear entirely rational, well behaved patterns of past financial theorists. They are seen for what they are: dynamic, unpredictable, and sometimes dangerous systems for transferring wealth and power, systems as important for us to understand as the wind, the rain and the flood. And floods- natural or manmade- need defenses”
(Mandelbrot 2004 p121)

For this reason, the use of fractal statistics would allow risk managers to comprehend the risky financial motion, and thus, be better prepared when these “inconvenient” outliers appear in the market. Therefore, this thesis will propose that a change in the assumptions behind risk models will lead to an improvement in risk management practice. To illustrate this point, it will be shown how a shift in the assumption from a normal market to a stable Paretian in the calculation of Value-at-Risk can lead to a better assessment of risk.

1.2 Value-at-Risk in a Stable Paretian Market

In the financial industry today, the most widely used measure to manage market risk is Value-at-Risk. However, in times of crisis, where is more necessary, it has proven to be inadequate. Within the framework of Chaos and the Science of Fractals is possible to find more accurate techniques to calculate VaR. However this will mean changing the current assumptions behind VaR models to account for the real behavior of financial series.

1.2.1 The rise of VaR

The decade of 1960’s experienced a phenomenal growth in trading activity caused by a massive increase in the range of financial instruments, which grew rapidly in trading volumes. With the deregulation of the financial sector, and the prompt advance of information...
technology, the financial environment became even more volatile. As a result, financial institutions were forced to find better ways of managing risks.

Gradually, new methods for calculating the losses at the firm wide level began to be part of the tools of professionals within financial institutions. This gave rise to the notion of Value-at-Risk, the worst loss over a target horizon that will not be exceeded with a given level of confidence. If $c$ is the confidence level, then VaR corresponds to the $1 - c$ lower tail level of the distribution of returns. Formally, it can be defined as follows: $P(\text{Loss} > \text{VaR}) = 1 - c$. The choice of confidence level and time horizon depends on the purpose of VaR. However, the confidence level is typically between 95% and 1%. Regulation (Basel I Accord) recommends a 99% confidence level in a 10 days horizon.

![Figure 3.1: Example of VaR at the 99% confidence level, which has become a standard choice in the financial industry (taken from Jorion 2007 p 19)](image)

The best-known method to calculate VaR is the one created by JP Morgan, and published through its Risk Metrics service. Developing this methodology took some years, but by the early 1990’s the main elements were in place and working around banks and other financial institutions. This model uses a variance-covariance technique, just as in Markowitz Portfolio Theory, to calculate a single number that quantifies the firm’s aggregate exposure to market

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96 Broadly speaking, the Value-at-Risk of a portfolio is the maximum loss that can be suffered in a certain period of time, during which the composition of the portfolio remains unchanged.
risk. This technique has become “the de facto standard risk measure used around the world today” (Bradley and Taqqu 2003 p 46). However, it has underperformed drastically during the last years, putting it in the critical eye of academics and professionals that have pointed its numerous limitations as a measure of risk.

1.2.2 The risk in Value-at-Risk

Despite the use of the variance-covariance method, this technique presents a serious drawback for the task of managing risk. The quantile estimate of the tails is underestimated for high confidence levels due to its assumption of the normal distribution of returns. This is particular worrisome because empirical evidence has demonstrated that price returns, especially in the limit of high frequency, are characterized by heavy tails and high peaks. Therefore, risk estimations based on conditions of normality are seriously biased. As McNew (1996) highlighted “VAR is a mixed blessing. Perhaps the biggest problem with VAR is the main assumption in conventional models, i.e. that portfolio returns are normally distributed (in Embrechets, Resnick and Samorodntsky 1998 p1).

In risk management, accurate prediction of the probability of an extreme movement is imperative. Because extreme events are related to the tails of the distribution, other methods have been developed to complement the measure of “Normal VaR”. Although the use of these techniques has lead to a considerable improvement in VaR models, “the majority of these [still] suffer from excessive VAR violations, implying an underestimation of market risk” (Kuester, Mittnik and Paolella 2006 p 83). These models have been developed in a “normal world” and thus lead to misleading results when applying them to real life situations.

As it was explained before, within the framework of Chaos Theory and The Science of Fractals, stable Paretian distributions are proposed as an alternative to model financial returns. The advantages of these distributions for risk management are the following:

1) Similar to normal distributions, they are also stable under addition and, thus, allow the generalization of the Central Limit theorem, which shows that if the finite variance assumption is dropped, the sum of independent identical distributions converges to a stable distribution.

2) Stable distributions provide a good approximation of observed data as they can accommodate fat tails and high peaks better. Thus, they are able to reflect the real

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97 For a detailed explanation please refer to Appendix III.
98 Between them, it is found the Expected Shortfall, Extreme Value Theory, Regular GARCH models, and more recently copulas.
99 The empirical evidence of this statement is also found in Kuester, Mittnik and Paolella (2006)
risk of large and abrupt changes that is not taken into account in a Gaussian market\textsuperscript{100}.

3) Risk managers working with stable Paretian distribution can estimate more accurately VaR values. In the case of a “Normal VaR” about 95% of the observations would be about two standard deviations of the mean, and about 99% would lie within three standard deviations. In the contrary, with stable distributions, three-sigma events may occur with a much larger probability. This is especially true for high quantiles of the distribution, which are associated with very rare but very damaging adverse market movements.

Researchers have demonstrated the above affirmation. Rachev, Schwartz and Khindanva (2003) present one of the most complete investigations on this topic\textsuperscript{101}. These authors compared the results of normal-VaR with stable Paretian-VaR, concluding the following:

1) The stable modeling generally results in a more conservative and accurate 99% VaR estimate than the one made with the normal distribution assumption. In fact, the normal distribution leads to overly optimistic forecasts of losses in the 99% quantile.

2) With respect to the 95% VAR estimation, the normal modeling is acceptable from a conservative point of view. The stable model underestimated the 95% VaR, but the estimate was actually closer to the true VAR than the normal estimate.

\textbf{Figure 3.1:} Representation of Rachev, Schwartz and Khindanva (2003) studies about normal-VaR with stable Paretian-VaR. The real results can be found in Appendix IV.

\textsuperscript{100} The fact that markets behave in this way, also implies that investors are not able to protect themselves from large losses by techniques such as “stop-loss” orders. In stable Paretian markets prices decline very rapidly, thus it is impossible to carry out “stop-loss” orders at intermediate prices.

\textsuperscript{101} For the sample information, and corresponding graphs of the investigation refer to the Appendix IV.
Harmantzis, Miao and Chien (2005) found similar results. For VaR estimation at a confidence level of 99% heavy tails models, such as the stable Pareto, produced “more accurate VaR estimates than non-heavy tailed models, especially for data that exhibit heavy tails” (Harmantzis, Miao and Chien 2005 p9). However, in the 95% confidence level, Gaussian models resulted in more accurate VaR results. Again, Ortobelli, Rachev and Fabozzi (2009) concluded the following: “The empirical evidence confirms that when the percentiles are below 5%, the stable Pareto model provides a greater ability to predict future losses than models with thinner tails” (Ortobelli, Rachev and Fabozzi 2009 p 16).

As it is observed, empirical evidence demonstrates that in the 99% quantile VaR estimates with stable Pareto distributions are more accurate than assuming normal distributions. This implies a significant improvement in risk management, as the risk in the extremes of the distributions can be measure more adequately. Therefore, by changing the assumption in risk model from Gaussian statistics to fractals statistics, it would be possible to control better risk, and even, ensure a safer financial system.

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102 The data series included four currency exchange rates: USD/Yen, Pound/USD, USD/Canadian, USD/Euro; and six stock market indices: S&P500 (US), FTSE100 (UK), Nikkei225 (Japan), DAX (Germany), CAC40 (France), TSE300 (Canada). The data covered a period from January 1990 to December 2003; DAX started in January 1991; CAC40 in March 1991, and Euro/USD began in January 1999.

103 The study of these authors was made in the MSCI WorldIndex, a stock market index of 1500 “world” stocks. The index includes stock from emerging markets.

104 A perspective on regulation is made on this point at the end of this section.
II. A NEW SET OF QUALITATIVE TOOLS FOR RISK MANAGERS

Besides the contributions that the mathematics of complexity can give to risk management, the theoretical foundations of Chaos Theory and the Science of Fractals leave also important lessons to the professionals in this area. After all, “although risk management involves quantitative methods, the subject itself rests on a foundation that is qualitative (…) In many ways the subject is like engineering: it uses sophisticated tools; but context and judgment are everything” (Fusaro 2008 p 78). The following are some lessons risk managers should learn form this new paradigm:

1) **Be the little ship in the big storm**

One of the main contributions of this theory is that it recognizes that volatility and turbulence will not cease to exist, and crisis and accidents will happen again. Indeed, the essence of understanding the fractal approach to the market is that it increases the awareness that many more bubbles, ruins and extreme outliers are still to come, something than conventional theory omits as a fact. Thus, it encourages managers to better understand risk, the changing dynamics, and consciously plan for the adverse outcomes to position themselves better than competitors. Within this framework, therefore, a risk manager not only needs to control current risk, but also anticipate circumstances so that even in bad times the organization is able to “float” in the “big storm”.

2) **Be aware of the butterflies**

Chaotic systems, such as financial markets, are sensitive to initial conditions and progress in a non-linear behavior. This was explained earlier by the butterfly effect introduced by Edward Lorenz. Risk managers should take into account this type of behavior, as it is characteristic of crucial financial phenomena. For instance, it explains that risk is endogenous in capital markets, meaning that risk is amplified within system. This becomes especially pronounced in times of crisis where spill over effects appear in the financial system. Since risk management systems are in place precisely to deal with such exceptional episodes, they must include in their analysis endogenous dynamics that may exuberate risk.

To do this it is important to: 1) Use non-linear models, as they represent dynamic behavior with multiple solutions\(^\text{105}\); and 2) complement mathematical models with Stress analysis\(^\text{106}\) or other qualitative tools. These techniques would allow having a better perspective in key

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\(^{105}\) This is possible with the use of the Monte Carlo Method with a Multifractal Brownian motion, for instance. In this case, risk models would be able to account for wild changes, fat tails, long-term dependence, concentration and discontinuity.

\(^{106}\) Stress Analysis measures the vulnerability to unlikely (but possible) scenarios.
variables, their interdependencies (multiple cause and effect relationships), and progression over a short period of time. Furthermore, they would give flexibility to the forecasts, and rapid adaptability in case of sudden changes. This is important as chaotic systems are continuously evolving, going through different transition phases and critical stages.

3) **Discover the fractals**

Chaos Theory breaks with traditional science by declaring that certain predictable patterns do exist in the financial system. In fact it argues that the nature of risk is universal, as only the scale changes. Consequently, daily, weekly, monthly or yearly, the same leptokurtic distributions are observed. Using chaos theory, therefore, allows risk managers to gain insight into the true nature of risk. If patterns are well analyzed it could be possible to draw conclusions about the causes behind these distributions such as market psychology, or changing external and internal conditions. As a result, although Chaos Theory is not a method for prediction, it does help understanding better the variables that drive risk factors.

Fractals also entail that complex dynamics arises from simple behavior. Therefore, risk managers should deal with complexity as simple as they can. However, that does not mean reductionist methods, but parsimonious methods that are appropriate to describe observed behaviour. Visualization of market risks, in this aspect, is essential. Risk managers should look for visual aid to understand the real dynamics of the market.

4) **Do not miss the elephant**

It is important for a risk manager to change from a micro perspective to a macro perspective. There is an old Hindu proverb about some blind men that tried to describe an elephant. They all explain it different as they all touch different parts of the body of the animal. Applied to risk management, this will mean: 1) adopting a “large scale” perspective to grasp the dimensions and global effects of risk; and 2) establishing risk control under a global focus. In other words, risk managers should determine appropriate firm wide policies that are coupled to a firm-wide risk management structure.

5) **Think outside the black box**

In chaotic circumstances, risk managers are forced to make decisions under conditions of extreme uncertainty. For this reason, they must appeal for “non-linear” thinking\(^{107}\), embrace the rules of disorder, and adapt to the new circumstances. As there are not magical “recipes” for treating this complex behaviour risk managers must develop the ability to address

\(^{107}\) “Discontinuous thinking is an invitation to consider the unlikely, if not the absurd” (Chorafas 1994 p 26)
problems in a more “creative way”. In other words, risk managers need insight and foresight to get significant results.

Conclusively, the new science of chaos and fractals not only gives risk managers better quantitative models to control risk, but also the adequate qualitative tools to cope with uncertainty. This would allow financial organizations, which are still dominated by thinking derived from the classical paradigm, to renew their risk management practice for the challenges imposed by today’s turbulent and unpredictable capital markets.
PERSPECTIVE ON REGULATION: What are the implications of changing the assumptions of VaR for regulation?

“Financial markets are risky, but in the carefully study of that concept of risk lies the knowledge of our world and hope of a quantitative control over it.

(Mandelbrot 2004 p 4)

The milestones of bank regulation on risk—taking behaviour are the 1988 Basel Accord (Basel I), the 1996 Amendment and the 2004 Basel Accord (Basel II). With Basel I, regulators established a minimum capital requirement for banks in order to ensure that these institutions hold certain amount of capital as a safety cushion in case of solvency and liquidity problems. The logic is that capital adequacy requirements can serve as a deterrent to unusual risk taking behaviour, if the amount of capital to set aside is tied to the risk undertaken. Accordingly, banks holding riskier assets must hold more capital as they have a higher probability of failure. It also distinguished between Tier 1 or core capital, and Tier 2 or supplementary capital. Banks must set aside an 8% capital charge, which must be covered by at least 50 percent of Tier 1 capital. The general 8% capital charge is multiplied by risk capital weight according to predetermined assets classes.

In 1996, due to the increase complexity of instruments and operations of banks, Basel I was amended to include risk-based capital requirements for market risks that banks incur in their trading accounts. In addition, it officially allowed banks to use either a standard approach or internal models based on VaR calculations. The first method was provided for the use of small and medium institutions that did not have complex technological infrastructure and

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106 Banks play a key role in the payment system. Because national governments guarantee the deposits of commercial banks, they have a direct interesting in assuring that banks will meet their obligations. This is one of the reasons why the amount of capital retained by a bank is regulated. By acting as a buffer, regulatory capital helps to private a burden that otherwise would be left in the hands of the government. Regulators also try to make sure that bank are well capitalized to avoid a “domino effect” where an individual institution failure would propagate to the rest of the financial sector.

109 Tier 1 capital is permanent and provides a high level of protection against losses. It includes proceeds from stock issues and disclosed reserves.

110 Tier 2 includes “perpetual securities, undisclosed reserves, subordinated debt with maturity longer than 5 years, and shares redeemable at the option of the issuer” (Reto 2003 p 58). Tier 2 capital is considered of lower quality than tier 1 because it may be redeemed eventually.

111 0% for cash, gold and OECD government bonds; 20% for OECD bonds from agencies and local (municipal) governments; 50% for uninsured mortgages; and 100% for corporate loans and claims issued by OECD non-governments, equity and property.

112 The standardized method computes capital charges separately for each market assigning percentage provisions for different exposures to equity, interest rate and currency risk. The total capital charge equals the sum of the market capital requirements. The main drawback of this approach is that it does not account for diversification effects.
expertise to calculate daily market risk exposures. Otherwise it was possible to create in-house models that require the approval of local regulators.

With this new approach, the capital requirement for general market risk\textsuperscript{113} is equal to the maximum of the current VaR number, and the average VaR over the previous 60 days multiplied by a factor between three and four. On the other hand, the capital charge for specific risk is equal to:

\[ C_s = A_t \times \max(\text{average VaR, current VaR}) + S_t \]

where \( A_t \) is a multiplication factor between 3 and 4\textsuperscript{114}, and \( S_t \) is the capital charge for specific risk. Both VaR values must be computed with a 10-day time horizon at a 99% confidence level using at least one year of data\textsuperscript{115}.

Nevertheless, Basel I was criticized because its rules were very simplistic, allowing banks to change their risk profile by simple modifying technical definition. As Dangl and Lehar (2002) said, it opened the opportunity for “regulatory capital arbitrage by intra bucket risk shifting i.e, increasing the risk of the bank’s assets without increasing the capital requirements”. (Dangl and Lehar 2002 p 3). For this reason, in 2004 a new Basel accord was agreed upon. Basel II Accord is based on three pillars: 1) Minimum regulatory requirements, which sets capital charges against credit risk, market risk and operational risk\textsuperscript{116}. Again, banks can use a standardized approach for calculating risks or use internal models, which normally include VAR approaches, subject to supervisor approval; 2) Supervisory review to strength the role of bank’s regulators; and 3) Market discipline, as this accord develops a set of disclosure recommendations encouraging banks to conduct their business in a safe, sound and efficient manner.

With the subprime crisis, in the meeting of the Basel Committee in March of 2008, the capital charge was again expanded to capture losses due to credit migrations and significant moves on credit spreads and equity prices. The consensus was then to add to the general market

\textsuperscript{113} The new capital requirements classify market risk into: 1) General market risk, which is the risk from changes in the overall market level, such as exchange rates, commodity prices, etc; and 2) Specific risk, which is the risk form changes in prices of a security because of reasons associated with the security’s issuer. Banks must calculate the capital requirement for both risks and then add them up.

\textsuperscript{114} The values depend on the accuracy of the VaR model in previous periods. Denote \( k \) as the number of times that actual losses exceed the VaR estimate over the last year (250 trading days). For regulators \( k \) can be in three zones. If \( k \) is in the green zone (\( k < 4 \)), \( k = 3 \). If it is in the yellow zone (\( 5 < k < 9 \)), \( k \) can range between 3 and 4; and if it is in the red zone (\( k > 10 \)), \( k = 4 \).

\textsuperscript{115} Presumably, the 10-day period corresponds to the time needed for regulators to detect problems and take corrective action. The choice of a 99 percent confidence level reflects the trade-off between the desire of regulators to ensure a safe and sound financial system and the adverse effect of capital requirements on bank profits.

\textsuperscript{116} Capital > Total Risk Charge = Credit-risk charge, market-risk charge and Operational risk charge. The 8% of risk-weighted assets was maintained
and specific risk an Incremental Risk Charge (IRC), which is calculated at a 99.9% confidence level over one-year horizon\textsuperscript{117}. The confidence level was set up to a higher level and the horizon was increased to be able to capture the losses that the 99%/10day VaR does not take into account. Therefore, the idea is that this new capital charge can include the risk of large and abrupt changes in the market.

Nevertheless, it seems that regulators are just adding sources of risk for the calculation of the capital requirement, as the solution for the underestimation of risk in current models. Indeed, discussion mainly evolves in the set of parameters such as the confidence level and time interval. However, they do not address the key problem: FAT TAILS in the distribution of returns. The focus of regulation, therefore, should be in those damaging outliers. As it was argued before, one remedy for this issue is to change Gaussian distributions to Stable Paretian, which have proved to be a better measure for high quantiles, such as 99% and 99.9%. Therefore, a mandatory change to the calculation of VaR in the framework of Stable Paretian hypothesis, will immediately improve the estimation of outliers. Consequently, regulators will be able to ensure that the internal models used to calculate risk are more accurate.

Furthermore, banks will be able to allocate capital more efficiently as the regulatory capital will be more in accordance with the risk taken. Danielsson, Hartman and Vries (1998) argued that the safety factor of three comes from the heavy tailed nature of the return of distributions that cannot be represented by the normality assumption. With this measure, regulators are trying to guarantee that institutions are solvent when crisis come. However, this also implies costs to financial institutions as an incorrect VaR leads to excessive regulatory capital. For this reason, an adequate assessment of risk can lead even to the reduction of this safety factor.

Conclusively, regulation is in a process of evolving, however it is tacking the wrong direction. Instead of looking for alternatives to the standard VaR statistic, is adding more layers of complexity. This could result in the wrong “remedy” for the risk-tacking disease of banks.

\textsuperscript{117} Regulators recognized that imposing a year horizon was unrealistic for many trading positions, thus it permits a liquidity horizon for each instrument, with the restriction that it cannot be shorter than three months. The definition of the model and parameters for the IRC is left largely to the banks.
CONCLUSION

Chaos Theory and the Science of fractals have already demonstrated the great progress it has brought to science. In physics, when scientists left the Newtonian vision of the world to observe its real complexity and roughness, they were able to have a better comprehension of natural systems. The advances that were done in this field, therefore, motivated other disciplines to take the same step forward. For instance, the inclusion of Chaos Theory in economics has allowed the exploration of economic phenomena with a more proper synthesis. Now, economists do not have to justify equilibrium, rationality and linearity; but instead they can address the intricate behaviour of economic reality.

Nevertheless, in both fields, it has been the effort of dissatisfied academics that have triggered a change in mainstream theories. In finance, this audacious work has been done for just a few, who have been ignored and have not received the proper attention. It seems like financists refuse to leave their old scientific methods of inquiry to embrace a new paradigm more in accordance with the new science. As a result, the ideas and method of Newton’s time, reflected in Neoclassical theory, continue to be deeply rooted in the financial industry, even so the world has change enormously.

This slow advance in financial theory has come with a very high price. Just in the last 20 years, it is possible to observe how financial crisis have augmented in number, size and value. Each one has struck the financial sector harder and in a more global scale. But in our current financial models, these events should have never happened. They were so improbable that they were just considered far far far outliers. Classical models simply fail to recognize the increasing complexity of financial markets, and consequently, they have lead financists to serious estimation errors.

For this reason, to be able to cope with the challenges of this new era, it is necessary to move away from the neoclassical approach to finance. This thesis proposes a fractal view of the market, as until now, it provides a more adequate perspective to understand financial behaviour. It recognizes its inefficiency and irrationality, and most important, it emphasizes it roughness. Consequently, this new paradigm would allow professionals in this area to work with the adequate vocabulary and method to address today’s capital markets. For risk managers, in particular, it would imply better models and analytical tools that can increase their awareness of the risky nature of markets. Thus, they would finally be able to receive clear warnings when trouble is ahead, and allow them to be better prepared. Perhaps, by
adopting Chaos Theory and the Science of Fractals in finance, the next crises can truly come with a “discounted” value for financial institutions.

"We live in an age of great events and little men, and if we are not to become slaves of our own systems or sink oppressed among the mechanisms we ourselves created, it will only be by the bold efforts of originality, by repeated experiment, and by the dispassionate consideration of the results sustained, and on unflinching thought."

Winston Churchill

March 31, 1934
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APPENDIX I. THE HURST EXPONENT

In the early XX century, H. E Hurst studied the record that the Egyptians had of the Nile River from 622 A.D to 1469 A.D\textsuperscript{118}. In his prolonged investigation, Hurst was able to discover that the Nile River followed non-periodic cycles. This meant that large overflows were more likely to be followed by other overflow. Just when an abrupt process changed the stream, the river switched to a period of low flow, until the cycle was altered again. This “persistence” in the successive yearly discharges of the Nile River revealed that some process in nature had “memory”, just as the Joseph effect in Mandelbrot’s story.

To give statistical meaning to his observations, Hurst decided to develop his own methodology. In this process, he found a more general form of Einstein equation\textsuperscript{119}, which could be applied to systems that were not independent. This resulted in the following law known as Hurst Empirical Law:

$$\frac{R}{S} = c \times n^H$$

The value of R/S is referred to the rescaled range analysis, where $c$ is a constant and $n$ refers to the R/S value for $x_1 \ldots x_r$. R/S has a mean of zero and it is expressed in terms of standard deviations. Essentially, the R/S value scales as we increase the time increment $n$, by a power-law value equal to $H$, known as the Hurst exponent\textsuperscript{120}.

Mandelbrot found that this exponent could be approximated by the following equation:

$$H = \log \left( \frac{R}{S} \right) / \log (n)$$

In short the R/S analysis is the range of partial sums of deviations of a time series from its mean, rescaled by its standard deviation. The methodology of the R/S consists in dividing the set of data into smaller intervals of time of equal size, to allow a separate analysis of each section. For each subperiod, the average value is calculated, and correspondingly, its departure from the mean. These values are aggregated into a single variable that includes those differences. The range is then defined as the maximum minus the minimum value of each subperiod, and later is normalized by dividing it with the standard deviation corresponding to each subperiod\textsuperscript{121}. In this way, the R/S for each subperiod $n$ is estimated. At the end, a linear regression of $\ln \left( \frac{R}{S} \right)$ against $\ln (n)$ will result in the Hurst coefficient.

\textsuperscript{118} This study was so extensive that Egyptians even called him, the “father of the Nile”.
\textsuperscript{119} This formula was used by Einstein to describe the distance cover by a random particle. It goes as follows: $R = T \Lambda 0.5$ where $R$ is the distance covered and $T$ the time index
\textsuperscript{120} This is the first connection of the Hurst phenomena with fractal geometry, as fractals scale also by a power law. More specifically in fractals the range increases according to a power, thus it is called power-law scaling.
\textsuperscript{121} Rescaling is important because it allows us to compare period that may be years apart. “By rescaling the data to zero mean and standard deviation of one, to allow diverse phenomena and time
Depending on the values that $H$ can take, it is possible to classify the dependence or independence of processes according to the following criteria:

- An $H$ equal to 0.5 implies a pure stochastic process, hence, variables will move independently. Nevertheless, it is important to notice that the R/S analysis does not require the process to be normally distributed. In fact, the R/S is a non-parametric measure that allows distributions such as the student t, gamma, between others.

- An $H$ that ranges between 0 and 0.5 describes an antipersistent behavior. Some theorists associate this with the concept of mean reversion, however this will involve a stable mean, which is not necessary in this kind of processes.

- An $H$ in the interval form 0.5 to 1 implies a persistent behavior. Persistent behavior is very characteristics of natural and even economic systems. For instance, Hurst observed that a variety of natural phenomena exhibit an $H$ of 0.7 and a standard deviation of 0.09.

R/S analysis has become an important tool for the development of fractal geometry. Although it was first introduced to describe long-term dependence of water levels in rivers and reservoirs now it has extended to the description of many processes in nature. Within the chaos framework, this analysis is important because it provides a sensitive method to describe long-term correlations in random processes.

Moreover, it can distinguish the characteristic of aperiodic cycles present in chaotic systems. Peters (1994) argues that the R/S analysis can discern cycles within cycles with the introduction of the $V$ statistic\footnote{This statistic was first introduced by Hurst (1951) to test for stability. Later, Peters use it to measure more precisely the cycle length, which works particularly well in the presence of noise.}. This statistic is defined as following:

$$V_r = (R/S)_r / s_{r}$$
If the process were independent, a plot of $V$ versus $\log(n)$ would be flat. In the contrary, if the process were persistent or antipersistent, it would have an upward or downward sloping, respectively. By observing this type of graphs, it is possible to notice the “breaks” in patterns, and hence estimate the length of the cycle (see Figure 2.5). Therefore, “the R/S analysis can not only find persistence, or long memory, in a time series, but can also estimate the length of periodic and non-period cycles\textsuperscript{123}. This makes it particular attractive for studying natural time series and, in particular market time series” (Peters 1994 p 102).

For instance, Peters (1994) analyzed the Dow Jones Industrial\textsuperscript{124} from January of 1988 to December of 1990 (102 years of daily data). In his research, Peters found that for the 20-day changes there was a persistent process of $H = 0.72$ and $E(H) = 0.62$, and for a five-day returns $H = 0.61$ and $E(H) = 0.58$. In both cases, the scaling process had a limit, more specifically, it occurred only for periods shorter than 1,000 trading days, approximately four years. This implied that the scaling process was not infinite, but instead it was a long process with finite memory in a non-periodic cycle. Peters' finding supports the idea that there is indeed local randomness and a global structure in the financial market.

\textsuperscript{123} The advantage of the R/S statistic is that it is robust with respect to additive noise, meaning the noise that does not affects the system, but it is actually a measurement problem
\textsuperscript{124} Published daily in The wall Street Journal since 1988. All holidays are removed from the time series.
APPENDIX II. MULTIFRACTAL ANALYSIS AND THE MULTIFRACTAL MODEL OF ASSET RETURNS (MMAR)

A multifractal process begins with a random function $F(\theta)$ where $\theta$ is “intrinsic time” or “trading time”. The possible function of $F(\theta)$ includes all the functions that model price variations, such as the Brownian Motion. In a separate step intrinsic trading time is selected as a scale invariant random function of the physical “clock time” $t$, thus the result is $\theta(t)$. From these two statistically independent random functions $F(\theta)$ and $\theta(t)$, where this last one is non-decreasing, a function $F(\theta(t)) = \varphi(t)$ is created. Mandelbrot explains it in a more comprehensive manner as the baby theorem:

Let $B(t)$ be a stochastic process, and $\theta(t)$ and increasing function of $t$. The following equation represents a compound or subordinated process:

$$X(t) = B(\theta(t))$$

where the index $t$ denotes clock time, and $\theta(t)$ trading time, representing the time deformation process. When the directing process is a martingale, fluctuations in trading time cause speeding up or slowing down of the process $X(t)$ without influencing its direction\(^{125}\). This allows fractional Brownian motion to proceed in both rapid but continuous changes and arbitrarily sharp discontinuities.

Multifractal analysis, which was also initially introduced to investigate turbulence in nature, has now been extended to financial time series. One of the most consistent proposals is known as the Multifractal Model of Asset Returns (MMAR). This model assumes that the log prices $\{X(t) = \ln P(t) - \ln P(0); 0 \leq t \leq T\}$ is a compound process satisfying the subsequent assumptions:

1. $X(t) = B_H(\theta(t))$ where $B_H$ is a Fractional Brownian Motion of exponent $H$ and $\theta(t)$ is a stochastic Multifractal trading time process. Because both, FBM and $\theta(t)$, are self-similar, self-similarity is also characteristic of a compound process.

2. $\theta(t)$ is the cumulative distribution function of a multifractal measure $\mu$ defined on $[0, T]$. It is important to highlight that trading time controls the tail properties of the process $X(t)$.

3. $B_H$ and $\theta(t)$ are independent.

\(^{125}\) Compounding can thus separate the direction and the size of price movements, and hence can be use to model the unobserved natural time-scale of economic series.
Multifractal measures can be obtained by iterating a procedure called a multiplicative cascade\textsuperscript{126}.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{multifractal_cascade}
\caption{Mandelbrot’s sample of a multifractal “cartoon” using the MMAR model. (Taken from Mandelbrot 2004 p 221)}
\end{figure}

The importance of these models resides on their characteristic properties. First, it combines several elements of previous research such as the fat tails in the unconditional distribution of returns, as in the Stable Paretian distributions. In fact, the MMAR improves stable processes by allowing a finite variance, and modeling the fluctuations in volatility. Second, volatility has long-memory in the absolute value of returns, the same as the FBM, and returns are even allowed to have a white spectrum or noise. And finally, the multifractal model contains volatility persistence at all frequencies\textsuperscript{127}. In short, the MMAR is a continuous time framework that includes both extreme variations and long memory volatility. It provides parsimony, due to the fact that the construction follows the same rule at each stage of the cascade, and flexibility to model a wide range of financial prices.

\textsuperscript{126} A detailed explanation is found in Calvet and Fisher (2002).
\textsuperscript{127} This means that volatility clustering is present at all time scales, corresponding to the evidence that economic factors such as technology shocks, business cycles, earning cycles, and liquidity shocks affect differently each scale.
APPENDIX II: CALCULATION OF VaR WITH VARIANCE-COVARIANCE METHOD

Markowitz' Mathematics

Markowitz method to calculate the return of a portfolio is based on the assumption of a linear relationship between risk and return, and considers the effects of diversification, using the standard deviation or variance as a measure for risk.

The portfolio return of the individual positions and the portfolio risk is the weighted risk of all individual assets and the covariance between those assets:

\[ R_p = \sum_{i=1}^{N} x_i r_i \]

\[ \sigma_p^2 = \sum_{i=1}^{N} x_i \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} x_i x_j \rho_{ij} \sigma_i \sigma_j \]

where N is the number of assets, \( r_i \) is the return of asset \( i \), and \( x_i \) is the weight of asset \( i \).

Delta-Normal method to calculate VaR

In the delta-normal method the same mathematics are used to calculate the portfolio VaR. Therefore, “if the positions are fixed over the selected horizon the portfolio rate of return is a linear combination of the returns of the underlying assets, where the weights are given by the relative amounts invested at the beginning of each period” (Jorion 2007 p 160).

The return of the portfolio from \( t \) to \( t+1 \) is:

\[ R_{(p,t+1)} = \sum_{i=1}^{N} w_i R_{i,t+1} \]

where N is the number of assets, \( R_{i,t+1} \) is the return of asset \( i \), and \( w_i \) is the its respective weight. Using matrix notation it can be represented as \( R_p = w' R \)

The portfolio expected return is

\[ E (R_p) = \mu_p = \sum_{i=1}^{N} w_i \mu_i \]

And the variance is

\[ V (R_p) = \sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j \]
In matrix notation \( \sigma_p^2 = W^T \Sigma W \)

As it is observed, just as in Markowitz portfolio theory, the volatility of VaR depends on the variances, covariance and the number of assets.

To translate the portfolio variance into a risk measure, it is assumed that all individual security returns are distributed normally. Then, it is possible to translate the confidence level \( c \) into a standard normal deviate \( \alpha \) such that the probability of observing a loss worse than \( -\alpha \) is \( c \). If \( W \) is the initial wealth, the portfolio VaR is defined as

\[
\text{Portfolio VaR} = \nabla \alpha \mathbf{R}_p = \alpha \sigma_p W
\]

This linear calculation of VaR is known as \textit{variance-covariance method} or \textit{covariance matrix approach}. 

Financial Data Series

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Taken from Rachev, Schwartz and Khindanva (2003) p 265

![DM/BP Empirical Density](image1)

![Yen/BP Empirical Density](image2)
Graphs are taken from Rachev, Schwartz and Khindanva (2003) p 269 - 278